Macroeconomic Dynamics, **13** (2009), 327–348. Printed in the United States of America.
doi:10.1017/S1365100508080073

LONGEVITY, RETIREMENT, AND CAPITAL ACCUMULATION IN A RECURSIVE MODEL WITH AN APPLICATION TO MANDATORY RETIREMENT

JIE ZHANG
National University of Singapore
and
University of Queensland

JUNSEN ZHANG
Chinese University of Hong Kong

This paper explores how retirement timing, together with life-cycle saving and human capital investment in children, responds to rising longevity in a recursive model with altruistic agents. We find that rising longevity raises the retirement age. If initial life expectancy is not too high, rising longevity also raises human capital investment in children and the saving rate. Through these channels, rising longevity can be conducive to long-run economic growth. A binding mandatory retirement age reduces human capital investment and the growth rate, raises the saving rate, and reduces welfare.

Keywords: Longevity, Growth, Life-Cycle Saving, Retirement

1. INTRODUCTION

Mortality rates have been declining dramatically in almost all countries for decades. The consequences of mortality decline on capital accumulation and economic growth have attracted a great deal of attention. See, for example, Ehrlich and Lui (1991), Auerbach and Kotlikoff (1992), Hu (1995), de la Croix and Licandro (1999), Boucekkine, de la Croix, and Licandro (2002), Bloom, Canning, and Graham (2003), Zhang, Zhang, and Lee (2001, 2003), Echevarría (2004), and Zhang and Zhang (2005). Most of these studies have ignored the response of retirement timing to declining mortality or rising longevity. By contrast, the numerous labor studies of retirement behavior surveyed by Lumsdaine and Mitchell (1999) have ignored the role of rising longevity.

We would like to thank an anonymous associate editor for giving us valuable comments and suggestions. All remaining errors or omissions are our own. Jie Zhang would like to thank Australian Research Council for its financial support (Grant DP0878752). Address correspondence to: Jie Zhang, School of Economics, University of Queensland, Brisbane, Qld 4072, Australia; e-mail: j.zhang@uq.edu.au.

© 2009 Cambridge University Press 1365-1005/09

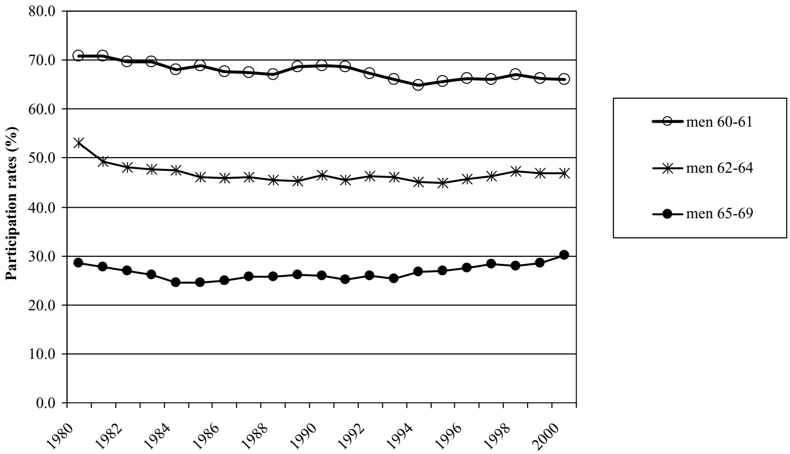

FIGURE 1a. U.S. male labor force participation rates.

To understand the relevance of the issue, let us take a look at the movement of the retirement age in the United States for the past few decades. Although it is hard to gauge it accurately, labor participation rates of people aged 60 or over have often been used in the literature as an approximate measure. It is well noted that male labor force participation rates have declined since the 1970s, but it is less noted that the decline has slowed down or even changed in the opposite direction recently. As illustrated in Figure 1a [based on U.S. Bureau of Labor Statistics (2001)], the participation rates of males aged 60 or over had declined before the late 1980s, but have been stabilized since and have been rising slightly in the late 1990s. Indeed, the increasing trend is very pronounced even since the early 1990s for the age group 65–69. In the meanwhile, the rise in the participation rates of females was more dramatic, driving up the total participation rate [see U.S. Bureau of Labor Statistics (2001); Hurd (1997)].[1] In the 1990s, the rise in female participation rates was small for those under age 45 (as the rates were already high), but large for the older groups as shown in Figure 1b. Female labor is now close to half of the total labor force [also see U.S. Census Bureau (2000, 2001)]. Although the converging participation rates by gender may largely reflect the diminishing gap in wages between men and women, the steady rises in the participation rates of both males and females aged 60 or over in the late 1990s need other explanations that may include mortality decline. As mortality is expected to fall further in coming decades, some observers, for example, Lee and Skinner (1999), have speculated that the age of retirement will go up; see also the statistical predictions of higher participation rates in 2008 for both males and females over age 55 by the U.S. Census Bureau (2000).

A related policy issue is the institution of the mandatory retirement age, which differs substantially between developing countries with low life expectancy (e.g., 50–55 in many African countries) and developed countries (e.g., 60–69 in North

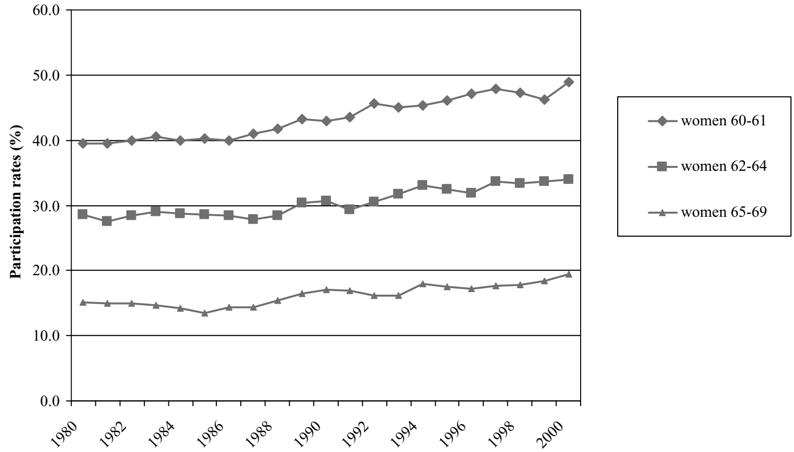

FIGURE 1b. U.S. female labor force participation rates.

America and West Europe). However, there appear to have been two trends in different directions concerning the mandatory retirement age in North America and Europe. One has been to encourage or regulate early retirement in Europe, whereas the other has been to raise or eliminate the mandatory retirement age in the United States (the 1986 Amendments of the Age Discrimination in Employment Act). It is interesting to know which trend is consistent with household optimizing behavior.

The body of existing studies on how retirement timing changes with rising longevity is relatively thin. Boucekkine et al. (2002) and Echevarría (2004) find that rising longevity raises the retirement age and time for schooling. In addition to the positive effect of longevity on the retirement age, Hu (1995) and Bloom et al. (2003) also find that rising longevity raises physical capital per person or the saving rate. The positive effects of longevity on schooling time and savings are consistent with the empirical evidence from cross-country data in Zhang and Zhang (2005), in the whole sample or the sample of countries with initial life expectancy in 1960 below 65. However, these effects of longevity become statistically insignificant for countries with initial life expectancy in 1960 above 65. All of these studies on longevity versus retirement timing have used the life-cycle model of consumption smoothing and ignored family structure consisting of different generations. In their framework, saving and education investment are all aimed at enhancing one's own income later. To the policy front, all these studies have ignored mandatory retirement and its consequence on capital accumulation.

In the present paper, we will investigate rising longevity and retirement timing by taking a different approach in a family setting. Our approach extends the dynastic family model by adding old-age consumption to motivate parents' own life-cycle saving, and exploits its parental altruism toward the welfare of children to motivate investment in children's education. We believe that the trade-off between life-cycle consumption and investment in children's education is a major concern of household decisions. In particular, whether eduction investment is made for oneself or for children differs in that the former can compensate the loss of one's own wage income from forgoing time for education but the latter cannot. This would intuitively lead to different scenarios for households to balance saving and education investment. It would be interesting to see whether our complementary approach can yield different or similar results. Indeed, Bloom et al. (2003, p. 326) point out that "the greatest worry is that actual savings decisions are made by households, not individuals, and that longevity and demographic effects operate through changes to family structure rather than individual consumption smoothing."

Another feature of a dynastic family model is that its decentralized equilibrium solution can be Pareto optimal as would be chosen by a social planner who considers the welfare of all generations, in contrast to a life-cycle model with overlapping generations of selfish agents whereby the equilibrium solution is usually not. This feature would help us determine the welfare consequence of mandatory retirement regulation, together with an analysis of how mandatory retirement affects capital accumulation.

In agreement with the life-cycle model, we find that a rise in life expectancy raises the retirement age. Unlike the life-cycle model, however, the positive effects of a rise in life expectancy on the saving rate and human capital investment are valid if initial life expectancy is not too high. Our simulation results also indicate: (i) the magnitude of these effects can be very substantial; (ii) these effects are mainly made when life expectancy rises from low to mid levels. Moreover, a binding mandatory retirement age reduces human capital investment and the growth rate, raises the saving rate, and reduces welfare. The magnitude of these distortions by mandatory retirement can be very large as well.

The remainder of the paper proceeds as follows. Section 2 focuses on retirement timing and saving in a simple life-cycle model of neoclassical growth for comparability with the literature. Section 3 provides the results in a dynastic family model of endogenous growth that also includes human capital (the core model). Section 4 applies the core model to the analysis of the impact of mandatory retirement. The last section provides some concluding remarks and policy suggestions.

2. SAVING AND RETIREMENT TIMING IN A SIMPLE LIFE-CYCLE MODEL

Before presenting the core model in the next section, it is useful to begin with a simple life-cycle model without human capital investment to be more comparable to the literature. The model has an infinite number of periods and overlapping generations of identical agents who live for three periods. The simplified time structure over lifetime, compared to the related studies in a continuous time framework, will allow us to incorporate human capital investment in children in a recursive model in the next section. In this setup, every agent has one unit of labor time endowment in each of the two periods in adulthood (the second and third periods of life). Survival from childhood to young adulthood is certain, but it is uncertain from the end of young adulthood to old adulthood at an exogenous rate $p \in (0, 1)$.[2] A rise in p reflects a fall in mortality or a rise in longevity. Every young adult has exactly one child.[3] Normalizing the size of the children population and the size of the young adult population to unity in every period, the size of the old-aged population is then p, that is, the fraction of the young adult population that survives to old age. The size of the total population is therefore $2 + p$.

In middle age, an agent supplies one unit of labor inelastically, and allocates wage income into middle-age consumption C and saving S for old-age consumption D. In old age, time is divided between leisure Z and labor $1 - Z$. We interprete leisure in old age as the time spent on retirement as in existing studies on retirement ages and longevity [see, e.g., Boucekkine et al. (2002); Bloom et al. (2003); Echevarría (2004)]. This compartmentalization of leisure at the old age is based on the fact that labor participation rates in prime ages are very high and stable [above 90% for men and around 80% for women between ages 25–50 in the United States according to the U.S. Census Bureau (2000)]. Realistically, leisure defined in this way becomes essential when an individual becomes old, because working with the same intensity is harder for old-aged than for middle-aged agents.[4]

There is an annuity market that channels saving to investment in physical capital. All saving S_t is traded for annuity income I_t conditional on survival to old age. Annuity firms invest S_t in the production sector and receive $R_{t+1}S_t$ in the next period where R is one plus the interest rate. Assume that the annuity market is competitive such that total annuity income paid out, pI_t, equals total revenue, $R_{t+1}S_t$. As a result, R_{t+1}/p is the rate of return on per unit annuity investment by households, I_t/S_t. This rate of return would depend inversely on the rate of survival if the interest factor were held constant. Intuitively, a rise in the survival rate raises the price of buying a given stream of annuitized income; namely, the same amount of saving can only buy less annuitized income at a higher survival rate.

The agent's budget constraints are given by

$$C_t = W_t - S_t, \tag{1}$$

$$D_{t+1} = R_{t+1}S_t/p + (1 - Z_{t+1})W_{t+1}, \tag{2}$$

where W is the wage rate per unit of labor.

The preference of the representative agent in this simple life-cycle model is assumed to be

$$U_t = \ln C_t + p(\ln D_{t+1} + \eta \ln Z_{t+1}), \quad \eta > 0, \tag{3}$$

where η is the taste for leisure in old age. The first-order conditions of maximizing the agent's utility in (3) with respect to saving and leisure are $1/C_t = R_{t+1}/D_{t+1}$ and $W_{t+1}/D_{t+1} = \eta/Z_{t+1}$, respectively.

Aggregate labor supply, denoted by E_t, includes 1 unit from young workers and $p(1-Z)$ from old workers, that is, $E_t = 1 + p(1 - Z_t)$.[5] The production function is given by

$$Y_t = BK_t^{\theta} E_t^{1-\theta}, \quad B > 0, \quad 0 < \theta < 1, \tag{4}$$

where K is physical capital, B the productivity coefficient, and θ the share parameter of capital. For simplicity, we assume full depreciation of physical capital within each period in production. Factors in production are compensated by their marginal products: $W_t = (1-\theta)Bk_t^{\theta}$ and $R_t = \theta Bk_t^{\theta-1}$, where $k_t = K_t/E_t$ is the ratio of physical capital to labor. Markets clear in this economy when $K_{t+1} = S_t$.

The equilibrium solution in this simple life-cycle model is found to be

$$Z_{t+1} = Z = \frac{\eta(\theta + p)}{p(1-\theta+\eta)} \text{ if } \eta < p(1-\theta)/\theta; \qquad Z = 1 \text{ if } \eta \geq p(1-\theta)/\theta, \tag{5}$$

$$S_t = \Gamma_s W_t, \qquad \Gamma_s = \frac{p\theta[1 + p(1-Z)]}{\theta(1+p)[1 + p(1-Z)] + p(1-\theta)(1-Z)}, \tag{6}$$

$$\frac{S_t}{Y_t} = \frac{p\theta(1-\theta)(1-\theta+\eta)}{\theta(1+p)(1-\theta+\eta) + (1+p\theta)[p(1-\theta) - \eta\theta]}. \tag{7}$$

From (5), leisure may have a corner solution when the taste for leisure is strong enough or when the rate of survival is too low. When it has an interior solution, the time spent on retirement is decreasing in the rate of survival, p. This reduction in the duration of retirement arises from opposing forces of rising longevity. On the one hand, there is a positive effect of rising longevity on the duration of retirement as seen in the numerator of the expression for Z in (5), because rising longevity increases the need for both consumption and leisure in old age. On the other hand, rising longevity tends to delay retirement by increasing saving (to be seen later) and by reducing the rate of return to annuity investment. The increase in saving tends to raise the wage rate, and thus induces later retirement. The decline in the rate of return to annuity investment also means that agents must save more or work longer to prevent old-age consumption from falling. (Note that the expected duration of retirement, pZ, can increase with p even though Z falls with p, as long as the elasticity of leisure in old age to a change in the rate of survival is less than 100%.) Obviously, when a rise in longevity reduces the duration of retirement, aggregate labor supply $1 + p(1-Z)$ increases further than the direct contribution of a rise in p to the number of old workers.

From (7), $\mathrm{d}(S/Y)/\mathrm{d}p$ is signed by $1 - p^2$ that is always positive. In other words, the ratio of total saving to total output is an increasing function of the rate of survival in this model. From (6), it can also be easily verified that the ratio of one's saving to one's middle-age labor income S_t/W_t is an increasing function of the rate of survival (even more so than the ratio of saving to output). Intuitively, expecting a longer life, middle-aged agents would have to save more for old-age consumption if they held old-age labor supply constant, as in the conventional overlapping generations models with all old-age lifetime spent on retirement, that is, $Z = 1$, and hence $S_t/W_t = p/(1 + p)$.

Overall, when longevity rises, the population of old-aged survivors, who receive the capital's share of output in the economy, becomes larger. Each of them has to expect a smaller amount of annuity income from saving, or to pay a higher price for a unit of annuity income, leading to an adverse effect on old-age consumption. To smooth life-cycle consumption and to keep a balance between consumption and leisure (assuming both are normal goods), agents save more in middle age and work longer in old age. This can be seen in the first-order conditions of the household problem where a rise in the rate of survival has a direct negative effect on old-age consumption D_{t+1} but no direct effect on both middle-age consumption C_t and old-age leisure Z_{t+1}.

The ratio of capital to labor evolves according to $k_{t+1} = \Gamma_s W_t/E_{t+1} = \Gamma_s(1 - \theta)Bk_t^{\theta}/[1+p(1-Z)]$ that converges to $k_\infty = \{\Gamma_s(1-\theta)B/[1+p(1-Z)]\}^{1/(1-\theta)}$. In this neoclassical growth model, there is a steady-state level of per capita output, $Y_\infty/(2 + p) = Bk_\infty^{\theta}[1 + p(1 - Z)]/(2 + p)$. There are three factors through which rising longevity may affect the steady-state level of per capita output. First, the positive response of saving, and hence the ratio of capital to labor, to rising longevity contributes to a higher steady-state level of per capita output. Second, the positive response of labor supply to rising longevity is conducive to higher per capita output. Third, a bigger population resulting from rising longevity lowers the steady-state level of per capita output. The sign of $\mathrm{d}k_\infty/\mathrm{d}p$ is the same as the sign of $\mathrm{d}(S/Y)/\mathrm{d}p$, which is positive as seen above. The sign of $\mathrm{d}\{[1 + p(1 - Z)]/(2 + p)\}/\mathrm{d}p$ is determined by $(1 - \eta)$, which is positive if $\eta < 1$. Thus, a sufficient condition for per capita output to respond positively to rising longevity is that the value of the taste for leisure is less than one, that is, valuing leisure less than consumption.

We sum up the results in this section here:

PROPOSITION 1. *For an interior solution in the life-cycle model, a rise in life expectancy reduces the duration of retirement, Z, and increases the saving rate, S_t/Y_t. Also, if agents value leisure less than consumption, then a rise in life expectancy increases the steady-state level of output per capita.*

The results in Proposition 1 generated from the simple model are similar to those in Bloom et al. (2003) that use a continuous time framework, suggesting that our simplification itself does not lead to any essential difference in results as they all belong to the life-cycle model. We now turn to the recursive model with altruistic parents.

3. THE CORE MODEL WITH HUMAN CAPITAL INVESTMENT IN CHILDREN

This section considers human capital investment in children motivated by parental altruism toward the welfare of children in a recursive dynastic model. This model enables us to investigate how retirement timing interacts with life-cycle saving and human capital investment in children in response to rising longevity. Because this model with two types of reproducible capital generates sustainable growth, we can also investigate how rising longevity affects the growth rate through influencing retirement timing, saving, and human capital investment.

In this core model, children learn to accumulate human capital H. Agents divide time in middle age between working $1 - M$ and educating their children M, and allocate wage income and time in old age in the same way as in Section 2. The agent's budget constraints become

$$C_t = (1 - M_t)W_t H_t - S_t, \tag{8}$$

$$D_{t+1} = R_{t+1}S_t/p + (1 - Z_{t+1})W_{t+1}H_t, \tag{9}$$

where W is the wage rate per unit of *effective* labor, rather than raw labor in Section 2.

The preference of the representative agent now also includes parental altruism toward children:

$$V_t = U_t + \alpha V_{t+1}, \quad 0 < \alpha < 1, \tag{10}$$

where V_t is the welfare of the agent, U_t is his period utility defined in (3), and V_{t+1} is the welfare of his child. Parameter α is the subjective discounting factor on the welfare of his child, or the taste for the welfare of his child. The recursive dynastic preference has an advantage over the conventional nonrecursive preference in overlapping generations models like the life-cycle model, in that the former can be applied to a social planner that considers the welfare of all generations. Consequently, with the dynastic preference, individual families and the social planner will choose the same retirement age in our model, a criterion to judge the welfare implication of mandatory retirement later.

As in Becker, Murphy, and Tamura (1990), the human capital of a child is obtained through his parent's time investment:

$$H_{t+1} = AM_t H_t, \quad A > 0, \tag{11}$$

where A is the productivity parameter in the education sector.

Aggregate labor supply includes $1 - M$ units from young workers and $p(1 - Z)$ from old workers. Effective labor supply is now given by $E_t = (1 - M_t)H_t + p(1 - Z_t)H_{t-1}$. The production function of the final good is the same as in (4).

Starting with initial human capital H_t, the representative agent's problem starting in period t corresponds to the following concave programming problem:

$$V(H_t; \Theta_t) = \max_{H_{t+1}, S_t, Z_{t+1}} \{\ln[W_t H_t - (H_{t+1}/A)W_t - S_t] + p\ln[R_{t+1}S_t/p + (1 - Z_{t+1})W_{t+1}H_t] + p\eta \ln Z_{t+1} + \alpha V(H_{t+1}; \Theta_{t+1})\} \tag{12}$$

taking the sequence of $\Theta_t \equiv (W_t, R_t, p)$ as given. In this setup, we have substituted (8) and (9) into (10) for middle-age consumption and old-age consumption, and used (11) to replace $M_t H_t$ with H_{t+1}/A. From the time-dependent recursive specification above to the fixed point in the value function, the standard Blackwell sufficient conditions apply here to apply the contraction mapping theorem and thus obtain a fixed point in the value function. We shall establish the existence and the uniqueness of the solution later.

Given the logarithmic utility function, we expect time-invariant proportional allocations of time and income for a given p, that is, constant M and Z, and constant fractions of middle-age income spent on middle-age consumption over time, denoted by $\Gamma_c \equiv C_t/[(1-M_t)W_t H_t]$, and on savings, by Γ_s. The first-order conditions can be derived routinely (see part A of the appendix). These conditions, the wage and interest rate equations, the budget constraints and the market clearing condition characterize the equilibrium of this economy. From these equilibrium conditions, we obtain the following relationships for the time allocation and proportional income allocation:

$$\Gamma_s = \frac{p[Z-\eta(1-Z)]}{(1+p)Z-\eta p(1-Z)}, \tag{13}$$

$$A\theta\eta(1-M)M + \theta\eta p(1-Z) = p(1-\theta)[Z-\eta(1-Z)], \tag{14}$$

$$(M-\alpha)[(1+p)Z-\eta p(1-Z)] = \alpha\eta p(1-Z)(1-M). \tag{15}$$

The equilibrium optimal allocation rules are implicitly determined in (13)–(15). Because there is no reduced form solution for the optimal allocation in general in this system of equations, it is essential to see whether there exists any equilibrium solution and whether the solution is unique before we proceed any further.

PROPOSITION 2. *There exists a unique solution for* (Γ_s, M, Z) *such that* $0 < \Gamma_s < 1$, $\alpha(1+p)/[1+p+p\eta(1-\alpha)] < M < 1$, *and* $Z > 0$. *In particular,* $M = \alpha$ *and* $\Gamma_s = p/(1+p)$ *at* $Z = 1$.

Proof. See part B of the appendix. ■

The existence of a unique equilibrium solution in this model allows us to investigate the complicated relationships among retirement timing, human capital investment in children, and life-cycle saving. Although the relationships are jointly determined and implicit in general, at the corner $Z = 1$ (because of a strong taste for leisure η) we can obtain reduced-form solutions for the saving rate and the fraction of time invested in children's human capital, that is, $\Gamma_s = p/(1+p)$ and $M = \alpha$ as in a conventional overlapping generations model with zero labor supply in old age. In this special case, a rise in longevity would clearly have a positive effect on the saving rate for consumption smoothing but no effect on the time allocation between work and education. The absence of any effect of longevity on human capital investment without retirement choice is also true in the special cases of Ehrlich and Lui (1991) and Zhang et al. (2001) when treating fertility as exogenously given. The reason why time investment in child education is unresponsive to rising longevity with a fixed retirement age is that parents would suffer a loss in their own wage income from investing more time in child education, and consequently a loss in their consumption in all life stages. As a result, the marginal utility of parental consumption would exceed the marginal utility of investment in children. This scenario differs from a life-cycle model like Boucekkine et al. (2002) and Echevarría (2004), in which the trade-off is between life-cycle saving and investment in one's own education investment for higher wage income later.

Outside the corner solution, the effect of rising longevity on the time allocation is given by:

PROPOSITION 3. *A rise in life expectancy raises the investment of time in child education*, M, *if* $M < 1/2$, *but reduces the duration of retirement*, Z, *for* $Z < 1$.

Proof. See part C of the appendix. ■

The positive response of human capital investment in the model with endogenous retirement is consistent with the estimated relationship between school enrollment and life expectancy when all countries in the whole sample or those with initial life expectancy below 65 are included [Barro and Sala-i-Martin (1995, Ch. 12); Zhang and Zhang (2005)]. However, this positive effect on human capital investment is only warranted in Proposition 3 when human capital investment in children takes less than half of parental time. It is thus possible that, when life expectancy is initially high and hence parents have already spent much time educating their children, any further rise in life expectancy may no longer have any significant effect on human capital investment. Indeed, the estimated coefficient on life expectancy in the education enrollment equation becomes statistically insignificant for the group of countries with initial life expectancy in 1960 above 65 in Zhang and Zhang (2005).

In this dynastic family model where altruistic parents invest in children's education, the duration of retirement of the interior solution still responds negatively to rising longevity as in the life-cycle model. In addition to the reasons given in the previous section, the delay of retirement can help compensate for the loss in parental labor income as a result of the increased time for child education driven by rising longevity.

The effect of longevity on life-cycle saving is given here:

PROPOSITION 4. *A rise in life expectancy raises the saving rate*, Γ_s, *at least if life expectancy is not too high.*

Proof. See part D of the appendix. ■

The reason for a positive response of life-cycle saving to rising longevity is similar to that mentioned in the life-cycle model, or in the conventional case with all old-age time spent on retirement. Unlike the life-cycle model, however, life-cycle consumption or saving is now traded off for investment in children's human capital rather than for parents' own human capital when allocating time between earning wage income and education. In the life-cycle model, the loss in wage income for one's own schooling is compensated by a higher wage rate for oneself, so that life-cycle saving and education investment can always increase with life expectancy. In the present model, by contrast, the loss in parental wage income for children's education is not compensated. Thus, it is possible in this recursive dynastic model that a further rise from a already high level of life expectancy may have little effect on both the saving rate and education investment.

We now examine how longevity affects growth. Suppose that the rate of survival rises unexpectedly in period 0 from p to $\tilde{p}$. Per capita output of the final good equals $Y_{-1}/(2+p)$ in time $t=-1$, and $Y_t/(2+\tilde{p})$ afterward for $t \geq 0$. From (4), the growth rate of per capita output from period -1 to 0, that is, $1+G_0 \equiv [Y_0/(2+\tilde{p})]/[Y_{-1}/(2+p)]$, can be written as

$$1+G_0 = \left(\frac{K_0}{K_{-1}}\right)^{\theta}\left(\frac{E_0}{E_{-1}}\right)^{1-\theta}\left(\frac{2+p}{2+\tilde{p}}\right). \tag{16}$$

The unexpected rise in life expectancy at $t=0$ does not affect decisions made previously such as K_{-1}, K_0, and E_{-1}. However, a higher $\tilde{p}$ at $t=0$ lowers the growth rate G_0 by accelerating population growth $(2+\tilde{p})/(2+p)$. It also has opposing impacts on initial growth by changing aggregate labor supply $E_0=(1-M_0)H_0+\tilde{p}(1-Z_0)H_{-1}$: a positive one directly through $\tilde{p}$, a negative one indirectly through raising M_0, and a positive one indirectly through reducing Z_0. In other words, a rise in life expectancy can promote growth in the initial period by increasing the labor supply from old-age population (directly or indirectly via reducing leisure), but it can harm initial growth by reducing labor supply from middle-age population (more time on education) and by accelerating population growth.

After the initial period (i.e. $t>0$), the population growth effect on per capita output growth disappears for a permanent rise in the rate of survival, whereas the factor E_t/E_{t-1} is equal to $A\tilde{M}$ for $t>0$. Moreover, for $t>0$, $H_{t-1}/E_t = 1/[(1-\tilde{M})A\tilde{M}+\tilde{p}(1-\tilde{Z})]$, and thus we have

$$k_t = \left[\frac{\tilde{\Gamma}_s(1-\tilde{M})(1-\theta)B}{(1-\tilde{M})A\tilde{M}+\tilde{p}(1-\tilde{Z})}\right]k_{t-1}^{\theta},$$

which converges monotonically to

$$k_{\infty} = \left[\frac{\tilde{\Gamma}_s(1-\tilde{M})(1-\theta)B}{(1-\tilde{M})A\tilde{M}+\tilde{p}(1-\tilde{Z})}\right]^{1/(1-\theta)}.$$

We may rewrite the growth equation for $t>0$ as

$$\ln(1+G_t) = \theta\ln[\tilde{\Gamma}_s(1-\tilde{M})(1-\theta)B] - \theta(1-\theta)\ln k_{t-1} + (1-\theta)\ln A + (1-\theta)\ln\tilde{M}. \tag{17}$$

During the transitional periods after period 0, the rise in life expectancy affects growth through the saving rate, the time spent on education, and the retirement age. These factors also determine the ratio of physical capital to effective labor, k_t, that affects growth. In the long run, the steady-state growth rate equals the growth rate of human capital accumulation AM that is increasing with longevity for $Z < 1$. In sum, we have the following.

PROPOSITION 5. *A rise in life expectancy initially has an ambiguous effect on the growth rate of per capita income. However, it promotes the steady-state growth of per capita income by increasing investment in human capital for* $Z < 1$.

Overall, the negative growth effects of rising longevity through population growth and labor supply are only a short-run phenomenon, and so is its positive growth effect through a reduction in leisure, despite that their impacts on the level of per capita output are long lasting as in Section 2. The positive growth effect of rising longevity through raising human capital investment will dominate in the long run.

What are the quantitative implications of rising longevity for the retirement age, human capital investment, the ratio of saving to output $\Gamma_s(1-M)AM(1-\theta)/[AM(1-M)+p(1-Z)]$, and the long-run growth rate? To answer these questions, we simulate the model and report the numerical results in Table 1. Parameterization in this table is mainly chosen such that the simulated figures are reasonably realistic, except that we adopt a widely used $\theta = 0.25$ for the physical capital's share in output, and that p varies in a wide range from 0.1 to 0.9. Parameter λ measures the productivity of human capital in old age relative to middle age, with $\lambda = 1$ standing for the case in which labor productivity does not decline with age. Also, suppose one period in this model equals 30 years.

According to Table 1, as the rate of survival rises from 0.1 to 0.9, the time spent on education rises from 25.1% to 30.5%, the time spent on retirement falls sharply

TABLE 1. Simulation results of rising longevity without mandatory retirement

$\alpha = 0.25,\ A = 6.0,\ B = 3.0,\ \theta = 0.25,\ \eta = 0.18,\ \lambda = 1.0$

Adult mortality (p)	Time on education (M, %)	Time in retirement (Z, %)	Saving rate (S/Y, %)	Annual growth ($G_\infty^{1/30}$, %)
0.1	25.11	73.95	6.28	1.38
0.2	25.65	47.04	9.44	1.45
0.3	26.33	38.13	11.33	1.54
0.4	27.05	33.68	12.57	1.63
0.5	27.78	31.00	13.44	1.72
0.6	28.50	29.21	14.07	1.80
0.7	29.19	27.93	14.53	1.89
0.8	29.85	26.95	14.88	1.96
0.9	30.50	26.19	15.14	2.03

from 74% to 26.2%, the ratio of saving to output rises from 6.3% to 15.1%, and the long-run growth rate rises from 1.38% to 2.03%. All these changes are substantial in magnitude, and are mainly made when life expectancy rises from low to mid levels. For further rises in life expectancy from a high level, these changes become much smaller.

4. APPLICATION TO MANDATORY RETIREMENT

Mandatory retirement has been instituted in many countries due to age-specific factors such as pension, health, and labor productivity. It may appear in different forms: a government program or a collective agreement between unions and employers. In this section, we only examine mandatory retirement in a simple case where labor productivity is age-specific: human capital in old age declines to λH_t with $\lambda \in (0, 1]$, with a depreciation factor $1 - \lambda$.

Suppose that the mandatory retirement age is $3-T$. The duration of mandatory retirement is $T \in [0, 1]$: A rise in T means a lower mandatory retirement age. Agents may choose early retirement by a fraction N of the old-age time $1 - T$ before reaching the mandatory retirement age. Thus, the total retirement time equals $Z = T+(1-T)N$, aggregate labor supply equals $1-M+p(1-T)(1-N)$, and aggregate effective labor at time t equals $E_t = (1 - M_t)H_t + p(1 - T)(1 - N_t)\lambda H_{t-1}$. An agent's budget constraint in middle age is the same as in Section 3, whereas the budget constraint in old age becomes

$$D_{t+1} = R_{t+1}S_t/p + (1 - T)(1 - N_{t+1})W_{t+1}\lambda H_t. \tag{18}$$

When the mandatory retirement age is not binding, that is $N > 0$, the solution for the model is obviously the same as in Section 3. We thus focus on the case where there is a binding mandatory retirement age, corresponding to $N = 0$, $Z = T$, $E_t = (1 - M_t)H_t + p(1 - T)\lambda H_{t-1}$, and $D_{t+1} = R_{t+1}S_t/p + (1 - T)W_{t+1}\lambda H_t$. The solution for the time invested in education is implicitly determined by

$$\begin{aligned}(1 + p)\theta[A(1 - M)M + p(1 - T)\lambda] + p\lambda(1 - T)(1 - \theta)\\ = \frac{\alpha p^2\lambda(1 - T)(1 - \theta)(1 - M)}{M - \alpha}.\end{aligned} \tag{19}$$

The saving rate is given by

$$\Gamma_s = \frac{p\theta[A(1 - M)M + p(1 - T)\lambda]}{(1 + p)\theta[A(1 - M)M + p(1 - T)\lambda] + p\lambda(1 - T)(1 - \theta)}. \tag{20}$$

The implicit solution shares the main features in Proposition 2:

PROPOSITION 6. *With a binding mandatory retirement age, there exists a unique interior solution for* (Γ_s, M) *such that* (*i*) $0 < \Gamma_s < 1$ *and* $\alpha < M < 1$ *for* $T < 1$; (*ii*) *for* $T = 1$, $\Gamma_s = p/(1 + p)$ *and* $M = \alpha$.

Proof. See part E of the appendix. ■

It can be easily verified that Propositions 3–5 apply in this case with a binding retirement age for $T < 1$. However, the strength of the effects of rising longevity should differ from the case without mandatory retirement. When old-age labor supply is fixed under a binding mandatory retirement age, a rise in longevity should raise the rate of return to education less than was in the previous section, so that human capital investment and hence the long-run growth rate should all rise less as well. Also, with the fixed old-age labor supply, old-age labor income will not respond to rising longevity so that the saving rate should rise to compensate for the loss in old-age annuity income more than in the case where old-age labor supply and labor income responded positively.

It is interesting to see how a rise in the duration of mandatory retirement T, or a decline in the mandatory retirement age $3 - T$, affects investment in human capital and saving.

PROPOSITION 7. *A lower mandatory retirement age* ($T < 1$), *when binding, reduces time spent on child education M but increases the saving rate Γ_s, at least if $M < 1/2$.*

Proof. See part F of the appendix. ∎

A lower binding mandatory retirement age reduces work time in old age. The consequent decline in old-age labor income needs a rise in the saving rate. The decline in work time in old age also reduces the rate of return to human capital, causing human capital investment to fall and middle-age labor supply to rise. The decline in human capital investment also implies:

PROPOSITION 8. *A lower mandatory retirement age* ($T < 1$), *when binding, reduces the long-run growth rate.*

In addition to the positive analysis, binding mandatory retirement represents additional constraints on individuals' choices, which must be welfare reducing in our model as its equilibrium solution without mandatory retirement would be chosen by a social planner.

What is the quantitative implication of rising longevity for the economy with mandatory retirement? How much is it different from the case without mandatory retirement? To answer these questions, we provide more simulation results in Tables 2 to 6, while maintaining the same parameterization for $(\alpha, A, B, \theta, \eta)$. We allow for two values for λ (1.0 or 0.8) and three values for T (0.0, 0.3, 0.7). As in Table 1, p varies in a wide range from 0.1 to 0.9.

In Tables 2 and 3, we use $\lambda = 1.0$ as in Table 1, i.e. labor productivity does not decline in old age, but allow mandatory retirement (T is equal to 0.3 and 0.7, respectively). Comparing Tables 1–3, some interesting points are worth noting. First, the signs of the effects of a rise in p are similar as those in Table 1. Second, the rises in human capital investment and the growth rate (the saving rate) with binding mandatory retirement are smaller (larger) than in Table 1. Third, a higher T (a lower mandatory retirement age), if binding, leads to less human capital

TABLE 2. Simulation results of rising longevity without mandatory retirement ($T = 0.3$)

$\alpha = 0.25,\ A = 6.0,\ B = 3.0,\ \theta = 0.25,\ \eta = 0.18,\ \lambda = 1.0$

Adult mortality (p)	Time on education (M, %)	Time in retirement (Z, %)	Saving rate (S/Y, %)	Annual growth ($G_\infty^{1/30}$, %)
0.1	25.11	73.95	6.28	1.38
0.2	25.65	47.04	9.44	1.45
0.3	26.33	38.13	11.33	1.54
0.4	27.05	33.68	12.57	1.63
0.5	27.78	31.00	13.44	1.72
0.6	28.48	30.00	14.15	1.80
0.7	29.13	30.00	14.75	1.88
0.8	29.77	30.00	15.22	1.95
0.9	30.38	30.00	15.58	2.02

investment, higher saving relative to income, and a lower long-run growth rate in the process of rising longevity. When the mandatory retirement age is set at the high level (a low T at 0.3) in Table 2, the distortions of mandatory retirement on the saving rate, education investment and the growth rate are small compared to Table 1. When the mandatory retirement age is set at the low level (a high T at 0.7) in Table 3, these distortions become large, particularly at high levels of life expectancy.

In Tables 4 to 6 with the value of λ being set to 0.8, we report simulations without and with mandatory retirement as in Tables 1 to 3. As λ becomes smaller (faster depreciation of human capital with age), saving tends to rise but both human capital investment and the long-run growth rate tend to fall. Also, the signs and

TABLE 3. Simulation results of rising longevity with mandatory retirement ($T = 0.7$)

$\alpha = 0.25,\ A = 6.0,\ B = 3.0,\ \theta = 0.25,\ \eta = 0.18,\ \lambda = 1.0$

Adult mortality (p)	Time on education (M, %)	Time in retirement (Z, %)	Saving rate (S/Y, %)	Annual growth ($G_\infty^{1/30}$, %)
0.1	25.11	73.95	6.28	1.38
0.2	25.42	70.00	10.55	1.42
0.3	25.80	70.00	13.75	1.47
0.4	26.23	70.00	16.17	1.52
0.5	26.69	70.00	18.07	1.58
0.6	27.16	70.00	19.59	1.64
0.7	27.64	70.00	20.81	1.70
0.8	28.11	70.00	21.82	1.76
0.9	28.57	70.00	22.65	1.81

TABLE 4. Simulation results of rising longevity without mandatory retirement

$\alpha = 0.25,\ A = 6.0,\ B = 3.0,\ \theta = 0.25,\ \eta = 0.18,\ \lambda = 0.8$

Adult mortality (p)	Time on education (M, %)	Time in retirement (Z, %)	Saving rate (S/Y, %)	Annual growth ($G_\infty^{1/30}$, %)
0.1	25.04	87.48	6.60	1.37
0.2	25.49	53.82	10.19	1.43
0.3	26.09	42.68	12.42	1.51
0.4	26.75	37.13	13.93	1.59
0.5	27.42	33.80	15.01	1.67
0.6	28.08	31.57	15.81	1.75
0.7	28.73	29.97	16.41	1.83
0.8	29.36	28.76	16.86	1.91
0.9	29.97	27.82	17.21	1.98

magnitudes of the impacts of higher longevity or a lower mandatory retirement age are similar to those in Tables 1–3.

A common feature across all the cases under mandatory retirement is that it becomes binding and distortionary when life expectancy reaches a certain high level. This indicates that the mandatory retirement age should change with life expectancy from time to time in order to avoid negative consequences on the economy. When it lies much below the retirement age chosen by the population, there would be pressure for change. This seems consistent with the elimination of mandatory retirement in North America. In the past two decades, mandatory retirement was prohibited in the public sector and cannot be set below 70 in other sectors in the United States, and was also banned in some provinces of Canada [see Gunderson and Riddell (1993)].

TABLE 5. Simulation results of rising longevity with mandatory retirement ($T = 0.3$)

$\alpha = 0.25,\ A = 6.0,\ B = 3.0,\ \theta = 0.25,\ \eta = 0.18,\ \lambda = 0.8$

Adult mortality (p)	Time on education (M, %)	Time in retirement (Z, %)	Saving rate (S/Y, %)	Annual growth ($G_\infty^{1/30}$, %)
0.1	25.04	87.48	6.60	1.37
0.2	25.49	53.82	10.19	1.43
0.3	26.09	42.68	12.42	1.51
0.4	26.75	37.13	13.93	1.59
0.5	27.42	33.80	15.01	1.67
0.6	28.08	31.57	15.81	1.75
0.7	28.73	30.00	16.41	1.83
0.8	29.33	30.00	17.00	1.90
0.9	29.91	30.00	17.47	1.97

TABLE 6. Simulation results of rising longevity with mandatory retirement ($T = 0.7$)

$\alpha = 0.25,\ A = 6.0,\ B = 3.0,\ \theta = 0.25,\ \eta = 0.18,\ \lambda = 0.8$				
Adult mortality (p)	Time on education (M, %)	Time in retirement (Z, %)	Saving rate (S/Y, %)	Annual growth ($G_\infty^{1/30}$, %)
0.1	25.04	87.48	6.60	1.37
0.2	25.34	70.00	10.89	1.41
0.3	25.67	70.00	14.33	1.45
0.4	26.04	70.00	16.99	1.50
0.5	26.44	70.00	19.11	1.55
0.6	26.86	70.00	20.81	1.60
0.7	27.28	70.00	22.21	1.66
0.8	27.70	70.00	23.37	1.71
0.9	28.12	70.00	24.34	1.76

The results in this paper will be more practically relevant if eliminating mandatory retirement in the real world has indeed led to a significant delay of retirement. So far, extensive evidence is unavailable in this regard. In a recent empirical study, Ashenfelter and Card (2002) have investigated the effect of the elimination of mandatory retirement on faculty retirement in some U.S. colleges and universities, using information on retirement flows over the 1986–1996 period. Interestingly, they have indeed found a significant rise in the fraction of older faculty after the elimination of mandatory retirement: The fraction of 70-year-olds who continued to work two years later were 8%–10% before the elimination and up to 40% after the elimination.

5. CONCLUDING REMARKS

In this paper, we have allowed the trade-off between leisure and labor in old age to investigate the effects of rising longevity on capital accumulation and retirement timing, and investigated the consequence of mandatory retirement. We began with a simple life-cycle model of neoclassical growth that generates essentially similar results as in continuous time life-cycle models: Rising longevity always raises the retirement age and investment in capital accumulation. We then used the simple life-cycle structure in a family setting and incorporated parental altruism toward the welfare of children to motivate human capital investment in children. Investment in own human capital, captured in the related life-cycle models, differs from investment in children's human capital, capture in our model, in that the former enhances own wage income but the latter does not. As a result, the loss in one's own wage income for spending time on education is compensated in the former case but not in the latter. With this in mind, we have shown why the positive effects of rising longevity on the saving rate and education investment are subject to an initial condition that longevity is not already too high, unlike the unrestricted positive effects in a pure life-cycle model. All of the models point to a delay of retirement in response to rising longevity, however.

When considering mandatory retirement, we found that binding mandatory retirement raises the saving rate but reduces human capital investment and the long-run growth rate. When longevity rises, the magnitudes of these distortions of mandatory retirement can become substantial if the mandatory retirement age is set much below the level that individual families would have chosen. Thus, the mandatory retirement age should adjust in tandem with life expectancy in order to mitigate its negative consequences on the economy and on agents' welfare. The recent elimination of mandatory retirement in some parts or occupations in North America seems in line with the household optimal behavior captured in this paper. These results also support proposals to raise the age at which individuals can begin drawing from public pension plans such as Social Security in the United States and similar plans in Europe and other developed countries.

NOTES

1. Participation rates of those over 60 fell in the 1970s for both males and females. Many observers attribute the decline to social security expansion at that time. See, for example, Blinder, Gordon, and Wise (1980), Boskin (1977), Hu (1979), Sheshinski (1978), and Kahn (1988).

2. According to Fries (1980) and Hammermesh (1985), it is not clear whether the maximum attainable age has increased despite rapid increases in life expectancies of adults in developed Western nations. By contrast, some studies find that this maximum age has risen [e.g., Wilmoth, Deegan, Lundstrom, and Horiuchi (2000)], a possibility that is not considered in this model. Also, death in fact may occur at any point in lifetime before or during retirement, which is not fully reflected in this simple overlapping generations model. Our abstraction from these dimensions does not change the substance of the results as will be seen later.

3. Like Hu (1995), Boucekkine et al. (2002), Bloom et al. (2003) and Echevarría (2004), we take fertility as exogenously given and focus instead on the labor-leisure trade-off in Section 2 and also on time spent on child education in Section 3. Beyond the scope of this paper, it would also be interesting to consider endogenous fertility as in Eckstein and Wolpin (1985) and Zhang et al. (2001).

4. The assumption of leisure to motivate retirement was also used in previous models of retirement [e.g., Gustman and Steinmeier (1986); Lazear (1986); Lumsdaine and Mitchell (1999)]; in particular, the preference for leisure increases exponentially with age, the spirit of which is shared in our paper. For tractability, they assumed a CES function and additively separable utilities derived from consumption and leisure, whereas, with two types of capital in our core model, we further simplify the utility function to a logarithmic function for tractability. In the meanwhile, we ignore other factors for retirement that have been considered in the literature, such as retirement income from pensions and social security in Stock and Wise (1990) and Lumsdaine, Stock, and Wise (1992), and firms' demand for old workers in Hammermesh (1993).

5. Labor productivity may decline with age, which does not affect the essence of the results here. We will explicitly introduce this element in Section 4.

REFERENCES

Ashenfelter, Orley and David Card (2002) Did the elimination of mandatory retirement affect faculty retirement? *American Economic Review* 92, 957–980.

Auerbach, Alan J. and Laurence J. Kotlikoff (1992) The impact of the demographic transition on capital formation. *Scandinavian Journal of Economics* 94, 281–295.

Barro, Robert J. and Xavier Sala-i-Martin (1995) *Economic Growth.* New York: McGraw-Hill.

Becker, Gary S., Kevin M. Murphy, and Robert Tamura (1990) Human capital, fertility, and economic growth. *Journal of Political Economy* 98, S12–S37.

Blinder, Alan S., Roger H. Gordon, and Donald E. Wise (1980). Reconsidering the work incentive effects of social security. *National Tax Journal* 34, 431–442.

Bloom, David E., David Canning, and Bryan Graham (2003) Longevity and life-cycle savings. *Scandinavian Journal of Economics* 105, 319–338.

Boskin, Michael J. (1977) Social security and retirement decisions. *Economic Inquiry* 15, 1–23.

Boucekkine, Raouf, David de la Croix, and Omar Licandro (2002) Vintage human capital, demographic trends, and endogenous growth. *Journal of Economic Theory* 104, 340–375.

de la Croix, David and Omar Licandro (1999) Life expectancy and endogenous growth. *Economics Letters* 65, 255–263.

Echevarría, Cruz A. (2004) Life expectancy, schooling time, retirement, and growth. *Economic Inquiry* 42, 602–617.

Eckstein, Zvi and Kenneth I. Wolpin (1985) Endogenous fertility and optimal population size. *Journal of Public Economics* 27, 93–106.

Ehrlich, Isaac and Francis T. Lui (1991) Intergenerational trade, longevity, and economic growth. *Journal of Political Economy* 99, 1029–1060.

Fries, J.F. (1980) Aging, natural death and the compression of morbidity. *New England Journal of Medicine* 303, 130–135.

Gunderson, Morley and W. Craig Riddell (1993) *Labor Market Economics*. Third Edition. New York: McGraw-Hill.

Gustman, Alan L. and Thomas L. Steinmeier (1986) A structural retirement model. *Econometrica* 54, 555–584.

Hammermesh, Daniel S. (1993) *Labor Demand*. Princeton, NJ: Princeton University Press.

Hammermesh, Daniel S. (1985) Expectation, life expectancy, and economic behavior. *Quarterly Journal of Economics* 100(2), 389–408.

Hu, Sheng-Cheng (1979) Social security, the supply of labor, and capital accumulation. *American Economic Review* 69, 274–283.

Hu, Sheng-Cheng (1995) Demographics, productivity growth and the macroeconomic equilibrium. *Economic Inquiry* 33(4), 592–610.

Hurd, Michael D. (1997) The economics of individual aging. In Mark R. Rosenzweig and Oded Stark (eds.), *Handbook of Population and Family Economics*, pp. 891–966. New York: Elsevier Science.

Kahn, James A. (1988) Social security, liquidity, and early retirement. *Journal of Public Economics* 35, 97–117.

Lazear, Edward P. (1986) Retirement from the labor force. In Orley Ashenfelter and Richard Layard (eds.), *Handbook of Labor Economics*, pp. 305–355. Amsterdam: North-Holland.

Lee, Ronald and Jonathan Skinner (1999) Will baby boomers bust the federal budget? *Journal of Economic Perspectives* 13(1), 117–140.

Lumsdaine, Robin L. and Olivia S. Mitchell (1999) New developments in the economic analysis of retirement. In Orley Ashenfelter and David Card (eds.), *Handbook of Labor Economics*, Vol. 3C, pp. 3261–3307. New York: Elsevier Science.

Lumsdaine, Robin L., James H. Stock, and David A. Wise (1992) Three models of retirement: Computational complexity versus predictive validity. In David Wise (ed.), *Topics in the Economics of Aging*, pp. 21–57. Chicago: University of Chicago Press.

Sheshinski, Eytan (1978) A model of social security and retirement decisions. *Journal of Public Economics* 10, 337–360.

Stock, James H. and David A. Wise (1990) Pensions, the option value of work and retirement. *Econometrica* 58, 1151–1180.

U.S. Census Bureau (2000, 2001) *Statistical Abstract of the United States*. Washington, DC: U.S. Government Printing Office.

U.S. Bureau of Labor Statistics (2001) *Civilian Labor Force Participation Rates by Sex, Age, Race, and Hispanic Origin*. Washington, DC: Office of Occupation Statistics and Employment Projections.

Wilmoth, J.R., L.J. Deegan, H. Lundstrom, and S. Horiuchi (2000). Increase of maximum life-span in Sweden, 1961–1999. *Science* 289, 2366–2368.

Zhang, Jie and Junsen Zhang (2005) The effect of life expectancy on fertility, saving, schooling and economic growth: Theory and evidence. *Scandinavian Journal of Economics* 107, 45–66.

Zhang, Junsen, Jie Zhang, and Ronald Lee (2001) Mortality decline and long-run economic growth. *Journal of Public Economics* 80, 485–507.

Zhang, Jie, Junsen Zhang, and Ronald Lee (2003) Rising longevity, education, savings, and growth. *Journal of Development of Economics* 70, 83–101.

APPENDIX

A. First-order conditions. Differentiate (12) with respect to H_t and H_{t+1}, respectively,

$$V'(H_t) = \frac{W_t}{C_t} + \frac{p(1 - Z_{t+1})W_{t+1}}{D_{t+1}},$$

$$\frac{W_t}{C_t A} = \alpha V'(H_{t+1}).$$

These two equations provide the Euler equation

$$\frac{W_t}{C_t A} = \frac{\alpha W_{t+1}}{C_{t+1}} + \frac{\alpha p(1 - Z_{t+2})W_{t+2}}{D_{t+2}}.$$

Differentiate (12) with respect to S_t and Z_{t+1}, respectively,

$$\frac{1}{C_t} = \frac{R_{t+1}}{D_{t+1}}.$$

$$\frac{W_{t+1}H_t}{D_{t+1}} = \frac{\eta}{Z_{t+1}}.$$

These equations equalize the marginal utility loss to the gain with regard to each of the variables. ■

B. Proof of Proposition 2. From (14) and (15), we have

$$\frac{p\eta(1-\alpha)M}{[1 + p + p\eta(1-\alpha)]M - \alpha(1 + p)} = \frac{\eta[A\theta M(1 - M) + p]}{p(1 - \theta + \eta)}.$$

Let the left side of this equation be $X_l(M)$ and the right side be $X_r(M)$, both as expressions of Z. For $M \in [0, 1]$, we have: $X_r' > 0$, $(=, >)$, if $M < 1/2$, $(=, >)$; $\min(X_r) = X_r(0) = X_r(1) = \eta/(1 - \theta + \eta)$; and $\max(X_r) = X_r(1/2) = [(A\eta\theta/4) + \eta p]/[p(1 - \theta + \eta)]$. Although X_r is single valued and continuous for $M \in [0, 1]$, $X_l(M)$ is discontinuous at $M = \alpha(1 + p)/[1 + p + p\eta(1 - \alpha)]$, declining from $X_l(0) = 0$ to $-\infty$ before M reaches this special point, jumping up to $+\infty$ at that point, and then declining to $X_l(1) = p\eta(1-\alpha)/[(1-\alpha)(1+p+p\eta)]$. For both $M \in [0, \alpha(1+p)/[1+p+p\eta(1-\alpha)])$ and $M \in (\alpha(1 + p)/[1 + p + p\eta(1 - \alpha)], 1]$, $X_l'(M) < 0$. Clearly, for $M \in [0, \alpha(1 + p)/[1 + p + p\eta(1 - \alpha)])$, there is no interior solution for M because X_r and X_l have no intersection given that $X_r > X_l$ in this range. For $M \in (\alpha(1+p)/[1+p+p\eta(1-\alpha)], 1)$, there must be a unique intersection of the two curves because (i) X_l falls monotonically from $+\infty$ to a level $X_l(1)$ below $X_r(1)$ and (ii) X_r is continuous and single valued. Denote this solution by $M^* \in (0, 1)$. Then we have $Z^* = X_r(M^*) > 0$.

To show $\Gamma_s \in (0, 1)$, we use (15) to obtain $M = \alpha(1+p)Z/[(1+p)Z - p\eta(1-Z)(1-\alpha)]$ or $1 - M = (1-\alpha)[(1+p)Z - p\eta(1-Z)]/[(1+p)Z - p\eta(1-Z)(1-\alpha)]$. Given that $M, Z > 0$, it follows that $(1+p)Z - p\eta(1-Z)(1-\alpha) > 0$. Given this and $1 - M > 0$, it follows that $(1+p)Z - p\eta(1-Z) > 0$. Substituting out M in (14) and (15), we have

$$\begin{aligned} &A\alpha\theta\eta(1-\alpha)(1+p)Z[(1+p)Z - p\eta(1-Z)] \\ &\quad = [(1+p)Z - p\eta(1-Z)(1-\alpha)]^2 p[(1-\theta)Z - \eta(1-Z)]. \end{aligned} \tag{A.1}$$

Since the left side of this equation is positive, $[(1-\theta)Z - \eta(1-Z)] > 0$ and hence $[Z - \eta(1-Z)] > 0$. Now $\Gamma_s > 0$ follows. Clearly, $\Gamma_s < 1$.

If leisure has a corner solution $Z = 1$, then the solution for Γ_s and M is obtained easily from (13) and (15). ■

C. Proof of Proposition 3. Consider the case with an interior solution for leisure, $Z < 1$. Rewrite the first equation in the proof of Proposition 2 as

$$[A\theta M(1-M)+p][M(1+p+p\eta-p\eta\alpha)-\alpha(1+p)] = p^2(1-\theta+\eta)(1-\alpha)M. \tag{A.2}$$

Differentiate it with respect to p: $M'(p) = X_n/X_d$ where $X_n = 2p(1-\theta+\eta)(1-\alpha)M - [1+p+p\eta(1-\alpha)]M + \alpha(1+p) - [(1+\eta-\alpha\eta)M - \alpha][A\theta M(1-M)+p]$ and $X_d = A\theta(1-2M)[(1+p+p\eta-\alpha\eta p)M - \alpha(1+p)] + [A\theta M(1-M)+p][1+p+p\eta(1-\alpha)] - p^2(1-\alpha)(1-\theta+\eta)$. It can be verified that $X_n > 0$. The sum of the second and third terms of X_d is positive under (A.2), whereas the first term is positive if $M < 1/2$ that is sufficient (not necessary) for $X_d > 0$ and hence $M'(p) > 0$.

Differentiate (A.1) with respect to p: $Z' = \tilde{X}_n/\tilde{X}_d$ where $\tilde{X}_n = Z[(1-\theta)Z - \eta(1-Z)]\{Z^2(1+p)(1-p) + p^2\eta Z(1-Z) + [Z - \eta(1-Z)(1-\alpha)][pZ(1+p)(1-p) + p^3\eta(1-Z) + 2p(1+p)^2 Z - 2p^2(1+p)\eta(1-Z)]\}$ and $\tilde{X}_d = p(1+p)\{Z\eta(1+\theta p) + [(1+p)Z - p\eta(1-Z)][Z(1+p+p\eta(1-\alpha)) + p\eta(1-\alpha)]\}$. It can be verified that $\tilde{X}_n > 0$ and that $\tilde{X}_d < 0$. ■

D. Proof of Proposition 4. Differentiating Γ_s with respect to p, Γ_s' is signed by $Z[Z - \eta(1-Z)] + p\eta Z'$. Using the expression of Z' in the proof of Proposition 3, it can be shown that Γ_s' is signed by

$$\begin{aligned} &Z(1+p)(1+p\theta-Z) + p^2 Z[Z - \eta(1-Z)] \\ &\quad + \eta p Z^2 + (1+p)[(1+p)Z - p\eta(1-Z)] \\ &\quad \times \{Z[1+p+p\eta(1-\alpha)] + p\eta(1-\alpha)\} - p\eta[Z - \eta(1-Z)(1-\alpha)] \\ &\quad \times \{Z(1+p)(1-p) + p^2\eta(1-Z) + 2(1+p)[(1+p)Z - p\eta(1-Z)]\}. \end{aligned}$$

Because $[Z - \eta(1-Z)] > 0$ and $[(1+p)Z - p\eta(1-Z)] > 0$, the first four terms above are positive and the last term is negative. If p is close to zero, then so will be the last term, whereas the rest of the terms will remain positive. Thus, the expression for Γ_s' is positive if p is not too high (sufficient but unnecessary). ■

E. Proof of Proposition 6. For $T < 1$, the proof parallels that of Proposition 2. For $T = 1$, the solution for Γ_s is obvious by (20), while the solution for M arises from (19) by multiplying both sides of (19) by $(M - \alpha)$ (note that $M = 1$ or $M = 0$ is suboptimal). ■

F. Proof of Proposition 7. Differentiating (19) with respect to T yields

$$M'(T) = \frac{[-\alpha p^2\lambda(1-\theta)(1-M) + (M-\alpha)p\lambda(1+p\theta)]}{p\lambda(1-T)[(1-\theta)(1+\alpha p)+\theta(1+p)] + A\theta(1+p)(1-M)M + (M-\alpha)A\theta(1+p)(1-2M)}.$$

The numerator of this expression is negative, which can be seen more clearly if using (19) to replace $-\alpha p^2\lambda(1-\theta)(1-M)$ by $-(M-\alpha)\theta(1+p)A(1-M)M/(1-T) - (M-\alpha)p\lambda(1+p\theta)$. The denominator is positive at least if $M < 1/2$. (Note from Proposition 6 that $M - \alpha > 0$ for $T < 1$.)

According to (20), the response of the saving rate to a rise in T, $\Gamma_s'(T)$, is signed by $(1-M)M + M'(T)(1-2M)$. Using the expression of $M'(T)$, $\Gamma_s'(T)$ is signed by $\{p\lambda(1-T)(1-\theta)(1+\alpha p) + (1+p)\theta[A(1-M)M + p\lambda(1-T)]\}/\{p\lambda(1-T)[(1-\theta)(1+\alpha p)+\theta(1+p)] + A\theta(1+p)(1-M)M + (M-\alpha)A\theta(1+p)(1-2M)\}$ that is positive under the same condition for $M'(T) < 0$. ■