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Life expectancy, human capital, social security and growth

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Abstract

We analyze the effects of changes in the mortality rate upon life expectancy, education, retirement age, human capital and growth in the presence of social security. We build a vintage growth, overlapping generations model in which individuals choose the length of education and the age of retirement, and where unfunded social security pensions depend on workers' past contributions. Social security has a positive effect on education, but pension benefits favor reductions in retirement age. The net effect is that starting from a benchmark case, higher life expectancies give rise to lower per capita GDP growth in the presence of social security as the share of the active population is reduced. In addition, higher social security contribution rates reduce the growth rate of per capita GDP.

JEL classification: O40; H55; J10
Keywords: Mortality Rate; Social Security; Growth

1. Introduction

The relationship between life expectancy at birth and per capita GDP growth has been studied both empirically and theoretically. Regarding the empirical evidence, the hypothesis that reductions in the mortality rate has a non monotonic relationship with per capita GDP growth is mostly supported. By using time series data, Rodriguez and Sachs (1999) find a positive effect from life expectancy to per capita GDP growth in Venezuela between 1970 and 1990. However,

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Malmberg (1994) finds a negative relationship in Sweden between 1950 and 1989. Analysis of cross section data also shows that this relationship is not monotonic. Preliminary data from Latin American and Caribbean countries indicate that the GDP growth rate is positively associated with life expectancy. [See World Health Organization (1999), Box 1.2, p. 9.] Barro and Sala i Martín (1995), using a sample of 97 countries, estimate that an increase in life expectancy of 13 years would increase the per capita growth rate by 1.4% per year. Zhang and Zhang (2005) show a clear, positive relationship, but at a diminishing rate. Still, other studies have found mixed evidence: increases in life expectancy have followed higher growth rates when life expectancies are low, but have followed lower growth rates when life expectancies are high. [See Zhang et al. (2003) and references therein].

Theoretical work mostly assumes that *human capital* accumulation is the engine of growth. Some studies conclude that the relationship is always positive, whereas others obtain an inverted *U* pattern. Among the former are Ehrlich and Lui (1991) and Hu (1999). In these models, higher life expectancy increases the length of time in which the return to human capital investment is received, thus allowing for higher rates of return, and, as a consequence, higher investment and per capita GDP growth rates.[1]

Still, other works have obtained an inverted *U* pattern between life expectancy and per capita GDP growth, which is consistent with the mixed empirical evidence mentioned above (both from historical and cross-section data). De la Croix and Licandro (1999) posit an economy where the effect of a reduction in the mortality rate upon the duration of education is such that the per capita GDP growth rate becomes higher for high mortality rates (as in underdeveloped countries), but lower for low mortality rates (as in industrialized countries). The same result is obtained in Boucekkine et al. (2002) under a setting in which there is an uncertain lifetime horizon and *endogenous retirement age*. In both papers, labor is the unique input in production, and the intuition behind the negatively sloped part is that the average human capital of the labor force becomes more *obsolete* as life expectancy increases.[2] Zhang and Zhang (2003) and Zhang et al. (2003) also obtain this result but by a different channel: not through own education time, but through expenditure on children's education.

Assuming exogenous growth, a second line of research has produced a number of articles dealing with the connections between population aging, a *pay-as-you-go* social security and the retirement age. One recurring topic in this literature is the effect of social security upon workers' voluntary retirement age. Along these lines, the available empirical evidence suggests that, at least for the US economy, social security is relevant for retirement age issues, even though there is not total agreement on the effect of changes in the payout from the social security program. [See, *e.g.*, Diamond and Gruber (1997) and Coile and Gruber (2000).]

In this article, we study to what extent introducing unfunded social security affects the relationship between life expectancy and per capita GDP growth, taking into account the social security impact on education and retirement age incentives. Our starting point is Boucekkine et al. (2002). Boucekkine et al. used an overlapping generations model with uncertain, finite lifetime

[1] Echevarría (2003) shows in an *exogenous* retirement model in which human capital investment depends positively on the number of working periods until the individual's retirement that increases in life expectancy give rise to higher GDP growth rates *only if* it is accompanied by simultaneous exogenous increments in the active period (*i.e.*, delays in the retirement age).

[2] Building on Boucekkine et al. (2002), but allowing for *physical* capital along with human capital in a certain lifetime horizon, Echevarría (2004) obtains the same relationship.

horizon. Fertility and mortality are exogenous, and individuals choose their optimal length of education and optimal retirement age, thereby influencing average human capital and the economy's growth rate in a vintage way. Our model extends this structure to include an unfunded social security system whose pension benefits depend on the contributions made by workers during their working period. According to this design, social security will influence not only *individual* decisions (namely, years of education and retirement age), but also *aggregate* variables such as economic growth.

Why might the inclusion of social security be of interest? In such a setup the return to human capital investment is not constrained to labor income while working, but also extends to pensions during retirement, which are in turn related to past wage earnings. Therefore, when individuals choose the optimal length of their education, they take into account not only the effect on future labor earnings, but also on future pension benefits. Additionally, voluntary retirement age will also depend on the incentives that the public pension system embeds. As a consequence, we find that social security will affect the size of the working population and the size of the aggregate human capital in the economy. This means that social security will influence the response of the economy's growth rate to changes in the mortality rate and the corresponding to changes in life expectancy.

This article is divided into two parts. In part one we solve analytically the individual problem (individuals and firms), the steady state per capita GDP growth rate and the social security budget balance. We also characterize the parameter space which determines the type of solution for the individuals' problem and prove the existence and the uniqueness of that solution. Furthermore, we prove the existence and uniqueness of the steady state per capita growth rate and social security balance for the case in which there are interior solutions for education and retirement age. In part two we numerically compare two scenarios: with and without unfunded social security. We are able to replicate the observed inverted *U* relationship between life expectancy and per capita growth in our scenario with social security. Our main finding is that introducing social security affects the incentives for education time and early retirement in such a way that the major force driving the negatively sloped part of that locus is the fall in the share of the working population (*i.e.*, workers), not the obsolescence of human capital among workers, as is the case when there is no social security.

The rest of the article is organized as follows. Section 2 introduces the economy: the demographic setup, the individual problem, the aggregate technology for production, the optimal length of education and the optimal retirement age, the economic aggregates, the social security balance and the balanced growth path. The numerical example and numerical results are contained in Section 3. Section 4 concludes. A mathematical Appendix contains the formal proofs.

2. The economy

Time is represented as a continuous variable. A unique good is produced which can be consumed, but not accumulated in the form of physical capital. Its price is normalized to one. The production technology uses human capital as the only production factor.

2.1. Demographics

At each instant of time τ, there is a continuum of cohorts born at different dates $t \leq \tau$. Individuals face an uncertain lifetime horizon with a positive, age increasing instantaneous mortality rate so that there exists a maximum attainable age.[3] The exogenous demographic structure in this closed economy is characterized by the *survival rate distribution*

$$m(a) = \frac{\alpha - e^{-\beta a}}{\alpha - 1}, \quad 0 \leq a \leq J, \quad \alpha > 1, \quad \beta < 0, \tag{1}$$

where $m(a) \equiv \text{Prob}(a_i \geq a)$ and a_i denotes an individual's death age. That is, Eq. (1) represents the probability of being alive at age a. J denotes the *maximum attainable age* which corresponds to a 0 survival probability and $m(J) \equiv 0$. This means

$$J \equiv \frac{-\ln\alpha}{\beta} > 0. \tag{2}$$

From Eq. (1), one can easily obtain the *life expectancy at birth* (to which we will refer as, simply, *life expectancy*) and denote by LE:

$$\text{LE} = \frac{1}{\beta} - \frac{\alpha \ln\alpha}{(\alpha - 1)\beta} > 0. \tag{3}$$

Note that all the previous expressions depend positively on both β and α. Population is assumed to grow at an exogenous, constant rate n, so that the *measure of births* at τ can be expressed as

$$\pi_\tau = \zeta e^{n\tau}, \quad \zeta > 0, \quad n \geq 0. \tag{4}$$

Using Eqs. (1), (2) and (3), one can obtain the *measure of population at* τ as the sum of measures of individuals born between τ and $\tau - J$ who have survived until τ,

$$P_\tau = \int_{\tau - J}^{\tau} \pi_t m(\tau - t) \mathrm{d}t = \zeta e^{n\tau} \kappa, \tag{5}$$

where κ is defined as

$$\kappa = \frac{\alpha\beta(1 - \alpha^{n/\beta}) + n(\alpha - 1)}{(\alpha - 1)(\beta + n)n} > 0. \tag{6}$$

From Eqs. (4) and (5) κ is interpreted as the *inverse of the crude birth rate*, the ratio of the measure of births to the measure of total population. Thus, an increase in life expectancy implies two extreme cases: i) a higher population growth rate n for a constant κ, or ii) a higher κ for a constant n.

From Eqs. (1), (4) and (5) it is possible to obtain the *probability density function* for age a as the ratio of the measure of individuals born at $\tau - a$ and alive at τ to total population at τ

$$f(a) = \frac{e^{-na}(e^{-\beta a} - \alpha)}{(1 - \alpha)\kappa}. \tag{7}$$

[3] This Section closely follows the first Section in Boucekkine et al. (2002). For further details, see Echevarría and Iza (2005), the working paper version of this article.

The *instantaneous mortality rate at age a*, denoted as $\xi(a)$, is defined as the ratio of the measure of individuals who die at age a to the measure of individuals surviving the age a. Thus, from Eq. (1) one obtains

$$\xi(a) = \frac{-\beta e^{-\beta a}}{\alpha - e^{-\beta a}} > 0, \tag{8}$$

which is strictly increasing in a. The observed mortality rates, however, fall during early childhood.[4] In spite of this, the simplicity of the demographic structure that we are using here makes this demographic structure highly attractive for theoretical models because it can replicate the observed age distributions in terms of life expectancy, maximum age and median age reasonably well. [See Table 1 in Echevarría and Iza (2005)].

The *mean age* and the *median age* of population, $\bar{a}$ and $\hat{a}$, respectively, can be obtained from Eq. (2) and (7) as $\bar{a} = \int_0^J af(a)\mathrm{d}a$ and $0.5 = \int_0^{\hat{a}} f(a)\mathrm{d}a$, or

$$\bar{a} = \frac{1}{(\alpha-1)\kappa}\left\{\frac{\alpha[1-(1+nJ)e^{-nJ}]}{n^2} - \frac{1-[J(\beta+n)+1]e^{-J(\beta+n)}}{(\beta+n)^2}\right\}, \tag{9}$$

and

$$0.5 = \frac{\alpha(\beta+n)-n}{(\alpha-1)\kappa(\beta+n)n} + \frac{1}{(\alpha-1)\kappa}\left[\frac{e^{-(\beta+n)\hat{a}}}{\beta+n} - \frac{\alpha}{n}e^{-n\hat{a}}\right]. \tag{10}$$

2.2. The individual problem

Denoting by t birth date and τ calendar time, the problem that an individual faces consists in finding the consumption path $c(t, \tau)$, the length of the education period $T(t)$ and the retirement age $R(t)$ which maximize his/her expected lifetime utility.[5] Both $T(t)$ and $R(t)$ should be interpreted in the sequel, of course, as planned. Workers' education has both individual consequences (higher labor income) and, as we will see later on, aggregate consequences (eventually, higher economic growth for the whole economy in the aggregate).

Instantaneous utility depends linearly on consumption.[6] Disutility from time spent on education and working (*i.e.*, other than in retirement) goes up as an individual's age $\tau - t$ increases. It also depends on average human capital in the economy $\bar{H}(t)$ at birth. Given that we will consider only steady state paths along which individual choices $R(t)$ and $T(t)$ remain time invariant, the marginal disutility of postponing retirement an additional period must be proportional to $\bar{H}(t)$. This is so because marginal utility out of the additional labor income obtained as a result of postponing retirement age of one period is also proportional to $\bar{H}(t)$. In sum, if devoting time to education and working means less leisure time, lifetime utility depends negatively on retirement age.

[4] For the U.S. case, for instance, at least since the 1940s the death rate attains its minimum at the 5–14 year age group, regardless of sex and race. See *Vital Statistics*, in U.S. Census Bureau, Statistical Abstract of the United States: 2004–2005, Table 96, p. 75.

[5] We assume that retirement obeys worker's leisure time preference only.

[6] We first tried the logarithmic instantaneous utility function to obtain an explicit solution for consumption in a previous version of this article. However, the non linearity of the model increased substantially, preventing us from obtaining analytical results and, additionally, giving rise to a multiplicity of solutions.

Thus, expected lifetime utility is given by

$$\int_t^{t+J} c(t,\tau)m(\tau-t)\mathrm{d}\tau-\frac{\bar{H}(t)}{\phi}\int_t^{t+R(t)}(\tau-t)m(\tau-t)\mathrm{d}\tau, \qquad \phi>0, \tag{11}$$

where $1/\phi$ stands for the disutility which both education and work time represent in terms of lost leisure.

The human capital with which this individual enters the labor market $h(t)$ depends on the number of periods devoted to education $T(t)$ and on the average human capital in the economy at the time of his/her birth, $\bar{H}(t)$. In particular,

$$h(t)=\mu\bar{H}(t)T(t), \qquad \mu>0. \tag{12}$$

There is, therefore, an externality in the production of human capital. It seems reasonable to assume that for a given education period, the human capital that the individual accumulates is higher when the knowledge in the economy as a whole is higher. This is, therefore, a public good that individuals enjoy but do not have to pay for. Similar mechanisms have been used previously in the *theoretical* literature. [See, *e.g.*, Zhang and Zhang (2003) and Lucas (1990)]. Also, *empirical* evidence largely supports the positive effect of class and school composition on individual students' educational attainment or the positive effect of local workers with longer education upon individual wages. [See Benabou (1993) and references therein].

During his/her active life the individual is paid a *gross* wage per unit of efficient labor equal to $\omega(\tau)$, and pays a (*pay-as-you-go*) social security tax at a constant rate $s\in(0,1)$. Thus, the net labor income obtained by this individual at time τ is equal to

$$w(t,\tau)=(1-s)\omega(\tau)h(t). \tag{13}$$

For simplicity, we assume that there is no depreciation of individual human capital while individuals remain on-the-job. Along these lines, Stokey and Rebelo (1995) claim that the largest source of depreciation of aggregate human capital comes from the fact that lifetimes are finite. Therefore, OLG models allow a more satisfactory treatment of this issue than infinite horizon representative agent models. This, in turn, raises a new problem: how human capital is transmitted from one generation to the next. In our model current generations learn from previous generations: they take advantage of the accumulated knowledge in the society when they are in their education period.

After retirement, the individual is paid a pension benefit equal to $b(t)$. The relationship between the social security contribution and the pension benefit is given by the *replacement rate* ϑ which we define (purely for analytical convenience) in terms of the average *net* wage income obtained during the active period,

$$b(t)=\vartheta(1-s)\bar{w}(t), \tag{14}$$

where

$$\bar{w}(t)\equiv\begin{cases}\displaystyle\int_{t+T(t)}^{t+R(t)}\frac{h(t)\omega(\tau)}{R(t)-T(t)}\mathrm{d}\tau\equiv h(t)\bar{\omega}(t), & \text{if } R(t)>T(t)\\ 0, & \text{if } R(t)=T(t)\end{cases} \tag{15}$$

denotes the *average gross wage income* earned throughout the same period. That is, $\bar{\omega}(t)\equiv[R(t)-T(t)]^{-1}\int_{t+T(t)}^{t+R(t)}\omega(\tau)\mathrm{d}\tau$ represents the *average gross wage per efficiency unit* earned while active.[7,8] This way, when making his/her optimal plan for education, the individual takes into account that more education time not only means higher wages while active, but also higher pension benefits while retired.

We assume that perfect annuity markets exist. Individuals do not save in physical capital (it does not exist), but in annuities. This kind of asset yields a return to its holder as long as he/she is alive. After his/her death, the property of the asset goes back to the insurance company that issued the asset. Thus, even if there were physical capital and individuals were not altruistic (so that they did not intend to leave bequests to their heirs), they would always prefer to save in annuities rather than in physical capital. The return to annuities would always be higher than that of physical capital because, in exchange, they would give back the annuities to the issuing company in case of death. This way the problem posed by unintended positive bequests is removed. Assuming that negative bequests are forbidden by law, individuals would also prefer to borrow in annuities. In exchange for cancelling out the debt in case of death, borrowers are forced to pay an extra return that compensates the lending company for the default risk in case of death.[9]

In short, the problem that the individual born at time t faces can be formally expressed as

$$\max_{\{c(t,\tau)\}_{\tau=t}^{\tau=J},T(t),R(t)} \int_t^{t+J} c(t,\tau)m(\tau-t)\mathrm{d}\tau - \frac{\bar{H}(t)}{\phi}\int_t^{t+R(t)}(\tau-t)m(\tau-t)\mathrm{d}\tau \tag{16}$$

subject to

$$\begin{cases} \int_t^{t+J} D(t,\tau)c(t,\tau)\mathrm{d}\tau = \int_{t+T(t)}^{t+R(t)} D(t,\tau)w(t,\tau)\mathrm{d}\tau + \int_{t+R(t)}^{t+J} D(t,\tau)b(t)\mathrm{d}\tau, \\ w(t,\tau) = (1-s)\omega(\tau)h(t), \\ h(t) = \mu\bar{H}(t)T(t), \\ b(t) = \vartheta(1-s)\bar{w}(t), \\ \bar{w}(t) = h(t)\bar{\omega}(t), \\ \bar{\omega}(t) \equiv [R(t)-T(t)]^{-1}\int_{t+T(t)}^{t+R(t)} \omega(\tau)\mathrm{d}\tau \\ R(t) \le J, \end{cases} \tag{17}$$

where $D(t, \tau)$ denotes the discount factor that applies between t and τ; *i.e.*, the price that an individual pays in t for one unit of consumption at time τ (contingent on being alive at that time).

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[7] In some countries pension benefits are linked to the worker's wage history: that is the case, among others, of the US [see Diamond and Gruber (1997)] and Spain [see Boldrin et al. (1997)]. In other countries, such as the UK, Holland or Sweden, pension benefits are the universal type. [See Miles (1999).] Zhang and Zhang (2003) assume a mixed setup.

[8] Pension benefits in our model are proportional to the average wage income earned while active for simplicity, but alternative assumptions could be made. For instance, the relationship between pension benefits and average wage income in the US is increasing, of course, but *concave*. [See Diamond and Gruber (1997), pp. 7–8.] In Spain, that relationship is *linear* (the proportion depending positively on the number of active periods), *but* subject to a minimum and a maximum. [See Jiménez-Martín and Sánchez (1999), pp. 49–50.] In both cases the incentives to early retirement are increased.

[9] This type of institution has repeatedly been used in theoretical models as a means to avoid accidental bequests. We will show below that, first, the instantaneous return of annuities is equal to the instantaneous probability of death. And, second, insurance companies issuing annuities obtain zero profits in equilibrium. Yaari (1965).

Upon substituting the second to fourth restrictions into the first one in Eq. (17), one obtains the following Lagrangian

$$\begin{aligned}\mathcal{L} = & \int_t^{t+J} c(t,\tau)m(\tau-t)\mathrm{d}\tau - \frac{\bar{H}(t)}{\phi}\int_t^{t+R(t)}(\tau-t)m(\tau-t)\mathrm{d}\tau - \lambda(t)\left\{\int_t^{t+J} D(t,\tau)c(t,\tau)\mathrm{d}\tau\right. \\ & \left.-\int_{t+T(t)}^{t+R(t)} D(t,\tau)(1-s)\omega(\tau)\mu\bar{H}(t)T(t)\mathrm{d}\tau - \int_{t+R(t)}^{t+J} D(t,\tau)\vartheta(1-s)\bar{\omega}(t)\mu\bar{H}(t)T(t)\mathrm{d}\tau\right\} \\ & -v(t)[R(t)-J], \end{aligned} \tag{18}$$

where $\bar{\omega}(t)\equiv[R(t)-T(t)]^{-1}\int_{t+T(t)}^{t+R(t)}\omega(\tau)\mathrm{d}\tau$, $\lambda(t)\geq 0$ denotes the Lagrange multiplier associated with the intertemporal budget constraint (the marginal utility of income), and $v(t)\geq 0$ is the Lagrange multiplier associated with the restriction that retirement age cannot exceed the maximum age limit J. Notice that Eq. (18) is linear in $c(t,\tau)$, so that we are implicitly assuming that consumption is non-negative.

The corresponding first order necessary conditions are given by

$$\frac{\partial\mathcal{L}}{\partial c(t,\tau)} = 0 \Leftrightarrow m(\tau-t) = \lambda(t)D(t,\tau), \tag{19}$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial R(t)} = 0 \Leftrightarrow & \frac{\bar{H}(t)}{\phi}R(t)m[R(t)] \\ = & \lambda(t)\mu\bar{H}(t)T(t)\omega[t+R(t)](1-s)D[t+R(t),t] - \lambda(t)\vartheta(1-s)\bar{\omega}(t)\mu\bar{H}(t)T(t) \\ & \times D[t+R(t),t] + \lambda(t)\int_{t+R(t)}^{t+J} D(t,\tau)\vartheta(1-s)\mu\bar{H}(t)T(t)\frac{\partial\bar{\omega}(t)}{\partial R(t)}\mathrm{d}\tau - v(t),\end{aligned} \tag{20}$$

where

$$\frac{\partial\bar{\omega}(t)}{\partial R(t)} = \frac{\omega[t+R(t)]\times[R(t)-T(t)] - \int_{t+T(t)}^{t+R(t)}\omega(\tau)\mathrm{d}\tau}{[R(t)-T(t)]^2}, \tag{21}$$

$$R(t)\leq J, \quad v(t)[R(t)-J] = 0, \quad v(t)\geq 0 \tag{22}$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial T(t)} = 0 \Leftrightarrow & 0 \\ = & \lambda(t)\int_{t+T(t)}^{t+R(t)} D(t,\tau)(1-s)\omega(\tau)\mu\bar{H}(t)\mathrm{d}\tau - \lambda(t)\mu\bar{H}(t)T(t)\omega[t+T(t)](1-s) \\ & \times D[t+T(t),t] + \lambda(t)\int_{t+R(t)}^{t+J}\vartheta(1-s)\bar{\omega}(t)\mu\bar{H}(t)D(t,\tau)\mathrm{d}\tau \\ & + \lambda(t)\int_{t+R(t)}^{t+J} D(t,\tau)\vartheta(1-s)\mu\bar{H}(t)T(t)\frac{\partial\bar{\omega}(t)}{\partial R(t)}\mathrm{d}\tau,\end{aligned} \tag{23}$$

where

$$\frac{\partial\bar{\omega}(t)}{\partial T(t)} = \frac{-\omega[t+T(t)]\times[R(t)-T(t)] + \int_{t+T(t)}^{t+R(t)}\omega(\tau)\mathrm{d}\tau}{[R(t)-T(t)]^2}. \tag{24}$$

In Eq. (20) we obtain that the marginal disutility of postponing retirement (in terms of lost leisure) must be equal to the marginal utility out of the augmented consumption that the additional income allows. Note that, i) $\lambda(t)$ represents the expected marginal utility out of income; ii) the sum of the terms that multiply $\lambda(t)$ is the marginal increase of the discounted future labor income; and iii) $v(t)$ is the expected marginal utility out of increasing the maximum lifetime horizon J (relevant if the restriction $R(t)\leq J$ is binding).

By definition, one has that $D(t,t)\equiv 1$ and that $m(t-t)=m(0)=1$; therefore, from Eq. (19) we obtain

$$\lambda(t)=1, \quad \text{and} \quad m(\tau-t)=D(t,\tau). \tag{25}$$

Given that the utility function is linear in $c(t,\tau)$, so is the Lagrangian Eq. (18): if no restrictions are imposed on the optimal consumption plan, the maximum of Eq. (18) is not well defined unless $m(\tau-t)=D(t,\tau)$. In other words, the budget constraint curve and the indifference curve coincide. If one imposed the non-negativity of $c(t,\tau)$ in an explicit manner, then Eq. (18) should be rewritten allowing for slackness variables. In that case, the equality between the discount factor $D(t,\tau)$ and the survival probability $m(\tau-t)$ would be obtained only for interior solutions.

Denoting the instantaneous return of an annuity by $r(x)$, from Eqs. (1) and (25) one obtains $\exp\left(-\int_t^\tau r(x)\mathrm{d}x\right)=[\alpha-e^{\beta(\tau-t)}](\alpha-1)^{-1}$. Upon differentiating both sides with respect to τ, we obtain that

$$r(\tau)=\frac{-\beta e^{-\beta(\tau-t)}}{\alpha-e^{-\beta(\tau-t)}}\equiv\xi(\tau-t). \tag{26}$$

In other words, the instantaneous rate of return at time τ for an individual born at t is identical to his/her instantaneous mortality rate defined in Eq. (8). An implication of Eq. (26) is that, assuming that insurance companies issuing annuities are risk neutral and perfectly competitive, they obtain zero profits (as expected). Denoting the stock of assets at time τ of an individual born at t by $W(t,\tau)$, the costs of the insurance company would be equal to $r(\tau)$ $W(t,\tau)$. But its revenues would be equal to $\xi(\tau-t)$ $W(t,\tau)$ because a fraction $\xi(\tau-t)$ of individuals of age $\tau-t$ would give back all their assets to the company on dying.

As we will see in Section 2.3, wages per efficiency unit ω are constant, so that the derivatives in Eqs. (21) and (24) are identically equal to zero, and $\bar{\omega}(t)=\omega$. Therefore, from Eqs. (19) and (25) we obtain that Eq. (20) can be rewritten as

$$-\frac{\bar{H}(t)}{\phi}R(t)m[R(t)]+\mu\bar{H}(t)T(t)\omega(1-s)m[R(t)]-\vartheta(1-s)\omega\mu\bar{H}(t)T(t)m[R(t)]-v(t)=0. \tag{27}$$

There are two open possibilities: i) If the optimal $R(t)$ is an *interior* solution, $R(t)<J$, then $v(t)=0$. From Eq. (27) one obtains $R(t)=\phi\mu T(t)(1-s)(1-\vartheta)\omega$. ii) If the optimal $R(t)$ is a *corner* solution, then $v(t)\geq 0$ and $R(t)=J$. Thus, we will have that

$$R(t)=\min\{\phi\mu T(t)(1-s)(1-\vartheta)\omega, J\}. \tag{28}$$

For the same reason, given Eqs. (25) and (28), Eq. (23) can be rewritten as

$$T(t)m[T(t)] = \int_{t+T(t)}^{t+\min\{\phi\mu T(t)(1-s)(1-\vartheta)\omega,J\}} m(\tau-t)\mathrm{d}\tau + \int_{t+\min\{\phi\mu T(t)(1-s)(1-\vartheta)\omega,J\}}^{t+J} \vartheta m(\tau-t)\mathrm{d}\tau. \tag{29}$$

Notice that Eq. (29) implies that $T(t) = T$ and, therefore, from Eq. (28) we have that $R(t) = R$. That is, *the optimal length of education and the optimal age of retirement are constant over time*. A key parameter for our discussion and one we will use repeatedly is $\eta \equiv \phi\mu$. If we define $\eta_0 \equiv 1/[(1-s)(1-\vartheta)\omega]$, the following two equations characterize T and R

$$R = \min\left\{\frac{\eta}{\eta_0}T, J\right\}, \tag{30}$$

and

$$Tm(T) = \int_T^{\min\left\{\frac{\eta}{\eta_0}T,J\right\}} m(\tau)\mathrm{d}\tau + \int_{\min\left\{\frac{\eta}{\eta_0}T,J\right\}}^{J} \vartheta m(\tau)\mathrm{d}\tau. \tag{31}$$

2.3. Technology of aggregate production

We assume that production technology is linear in human capital,

$$Y(\tau) = \omega H(\tau), \qquad \omega > 0, \tag{32}$$

where $Y(\tau)$ denotes aggregate production and $H(\tau)$ aggregate human capital at τ. The latter is equal to the sum of individual stocks of human capital across *workers* of different ages (born at different t's, but active at τ). This is, in sum, a *vintage* model as explained in detail in Section 2.5. Therefore not only is the time τ at which human capital is measured relevant, but so is the length of education T, retirement age R and age distribution of the population. Marginal productivity of human capital ω is constant and equals the (gross) wage per unit of efficiency. The parallelism with AK technologies in which production is proportional to the stock of aggregate capital in equilibrium is obvious. [See, among others, Barro and Sala i Martín (1995) for details.]

2.4. School and retirement age in equilibrium

In this subsection we characterize the optimal length of education and the optimal age of retirement. We will make the following distinction: *interior* solutions ($0 < T < R < J$) and *corner* solutions. These, in turn, can be of two types: $0 < T < R = J$, so that *planned* retirement age is given by the maximum lifetime horizon, and the length of education is equal to the *upper bound for interior solutions for* T which we will characterize later on. And $0 = T = R < J$, so that individuals choose neither to invest in human capital nor to enter the labor market. If so, both labor income and pension benefits are zero. This is possible given our assumption of linear utility from consumption.

2.4.1. Interior solution: $0<T<R<J$

In this subsection we find the conditions upon parameter η for the existence and uniqueness of an interior solution. If the solution is interior, from Eq. (30) we obtain that

$$R=\frac{\eta}{\eta_0}T, \tag{33}$$

where a *necessary* condition that η and η_0 (or, equivalently, η, s, ϑ and ω) must satisfy for $R>T>0$ is that $\eta>\eta_0$. Intuitively, given the definitions of η and η_0, for individuals to devote a part of their lifetimes to education *and* a part to active work one needs: i) high gross wages ω, ii) low social security contribution rates s, iii) high productivity of investment in human capital μ, iv) low disutility of time not devoted to leisure ϕ, and v) low replacement rates ϑ.

Note that if the solution is interior, R is proportional to T and, in particular, $R=\eta(1-\vartheta)(1-s)\omega T$. Therefore, for a given ϑ, the discouraging effect that a higher s has upon T is enlarged when we look at the effect upon R. This point will be relevant when we carry out our numerical exercise in Section 3.

Assuming that $\eta>\eta_0$, from Eqs. (30), (31) and (33) the optimal T is given by

$$Tm(T)=G_1(T)+G_2(T), \tag{34}$$

where we have defined

$$G_1(T)\equiv\int_T^{\frac{\eta}{\eta_0}T}m(\tau)\mathrm{d}\tau, \quad \text{and} \quad G_2(T)\equiv\vartheta\int_{\frac{\eta}{\eta_0}T}^{J}m(\tau)\mathrm{d}\tau. \tag{35}$$

In other words, for T to be optimal the cost of an additional education period (in terms of foregone wages) must be equal to the increment in the sum of expected discounted future wages *plus* expected pension benefits as a result of that additional learning period.

Besides, following the case of an interior solution so that $R<J$, Eq. (33) implies that

$$T<\frac{J\eta_0}{\eta}\equiv T_{\max}(\eta)<J. \tag{36}$$

There is, therefore, an *upper bound for the optimal T* (not only for interior solutions, but – as we will see – also for corner solutions in which $R=J$ and $T\equiv T_{\max}$).

Introducing unfunded social security with positive ϑ and s changes the lower bound of η for interior solutions. [See Boucekkine et al. (2002), Lemma 2.3, p. 350, who obtain $\eta>2$ as a necessary condition for interior solution]. Assuming $\omega=1$, one has that $\eta_0=[(1-s)(1-\vartheta)]^{-1}$ need not be higher than or equal to 2. For instance, the productivity of education time in the production of individual human capital need not be so high as to induce individuals to spend a fraction of their lifetimes on accumulating knowledge. Why? Because in the presence of a social security system like ours (unfunded and whose pension benefits depend positively on earned labor income in the past) pension benefits represent an additional incentive to the wage income obtained during the active period. In other words, pension benefits reduce the depreciation of human capital that a finite active period represents for workers.

The product $Tm(T)$ must be equal to the sum of $G_1(T)\equiv\int_T^{\frac{\eta}{\eta_0}T}m(\tau)\mathrm{d}\tau$ plus $G_2(T)\equiv\vartheta\int_{\frac{\eta}{\eta_0}T}^{J}m(\tau)\mathrm{d}\tau$. Therefore, $G_1(T)$ need not be so high as when there is no social security, the optimal T condition being given in that case by $Tm(T)=G_1(T)$.

If there were no social security, and given that $m(x)$ is strictly decreasing, one would have that

$$\left(\frac{\eta}{\eta_0}-1\right)Tm(\eta T/\eta_0) < \underbrace{\int_T^{\frac{\eta T}{\eta_0}} m(x)\mathrm{d}x}_{G_1(T)} < \left(\frac{\eta}{\eta_0}-1\right)Tm(T) \tag{37}$$

If the solution for T were interior, from the second of the two inequalities in (37) and (34) one would get

$$\int_T^{\frac{\eta T}{\eta_0}} m(x)\mathrm{d}x < \left(\frac{\eta}{\eta_0}-1\right)\left[\int_T^{\frac{\eta T}{\eta_0}} m(x)\mathrm{d}x + \vartheta\int_{\frac{\eta T}{\eta_0}}^{J} m(x)\mathrm{d}x\right]$$
$$\Rightarrow \eta > \left[\frac{2\int_T^{\frac{\eta T}{\eta_0}} m(x)\mathrm{d}x + \vartheta\int_{\frac{\eta T}{\eta_0}}^{J} m(x)\mathrm{d}x}{\int_T^{\frac{\eta T}{\eta_0}} m(x)\mathrm{d}x + \vartheta\int_{\frac{\eta T}{\eta_0}}^{J} m(x)\mathrm{d}x}\right]\eta_0.$$

If we assume that $\vartheta=0$, $\omega=1$ and $s=0$, we will have that $\eta>2\eta_0=2$, the same condition that Boucekkine et al. (2002) obtain. [See Boucekkine et al., 2002, Proposition 2.1, p. 350.]

To analyze the existence and uniqueness of the *interior* solution, we first define the following auxiliary continuous function in x and η:

$$M(x,\eta)\equiv(\alpha-1)[xm(x)-G_1(x)-G_2(x)]. \tag{38}$$

From Eqs. (34), (35) and (38) one has that T is an interior solution if and only if $x=T$ is a root of Eq. (38). Therefore, we will be able to discuss the existence and uniqueness of the interior solution upon studying the properties of $M(x,\eta)$.

Our strategy will be as follows. First, we will prove that $M(x,\eta)$ is negative in the origin $x=0$ [Lemma 1] and positive at $x=T_{\max}>0$ for an interval of values of η [Lemma 2]. The continuity of $M(x,\eta)$ will assure us that there *exists* at least one $x\in(0,T_{\max}(\eta))$ for which $M(x,\eta)=0$. [Proposition 1]. And, second, we will give sufficient and necessary conditions on η for such an x to be *unique*. [Propositions 2 and 3, respectively]. The proofs of the analytical results are only sketched in the Appendix because they all involve just basic algebra. For more detailed proofs, the reader is referred to Echevarría and Iza (2005).

Lemma 1. *Assume $\beta<0$ and $\alpha>1$: if $\vartheta>0$, then $M(0,\eta)<0$.*

Proof. See Appendix. □

We define the following auxiliary function $K(\eta)=M(T\max(\eta),\eta)$. From Eqs. (2), (36) and (38) it can be shown that

$$K(\eta)=\frac{1}{\beta}\left\{\frac{\eta_0}{\eta}\left(\alpha^{\frac{\eta_0}{\eta}}-2\alpha\right)\ln\alpha+\alpha(\ln\alpha-1)+\alpha^{\frac{\eta_0}{\eta}}\right\}, \tag{39}$$

which is continuous for $\eta>0$.

Lemma 2. *Assume $\beta<0$, $\alpha>1$ and $\eta_0>0$: then i) $K(\eta)$ is continuous for $\eta>0$; ii) if $\eta=\eta_0$, then $K(\eta)=0$; iii) there is a unique $\eta^*>2\eta_0$ such that $K(\eta^*)=0$; and iv) $K(\eta)>0$ if and only if $\eta\in(\eta_0,\eta^*)$.*

Proof. See Appendix. □

Proposition 1. *Existence. Sufficiency. Assume* $\beta<0$, $\alpha>1$, $\vartheta>0$ *and* $\eta_0>0$. *If* $\eta_0<\eta<\eta^*$, *then there is at least one interior solution for T and R which satisfies Eqs.(33) and (34), and for which* $0<T<R<J$, $T<T_{max}$.

Proof. See Appendix. □

Proposition 2. *Uniqueness. Sufficiency. Assume* $\beta<0$, $\alpha>1$, $\vartheta>0$ *and* $\eta_0>0$. *If* $\eta_0<\eta<\eta^*$, *then there is a unique interior solution for T and R which satisfies Eqs.(33) and (34), and such that* $0<T<R<J$, $T<T_{max}$.

Proof. See Appendix. □

Proposition 3. *Uniqueness. Necessity. Assume* $\beta<0$, $\alpha>1$, $\vartheta>0$ *and* $\eta_0>0$: *if the unique solution for T and R is interior [i.e.,* $0<T<R<J$, $T<T_{max}$, *and satisfies Eqs. (33) and (34)], then it must be the case that* $\eta_0<\eta<\eta^*$.

Proof. See Appendix. □

2.4.2. Corner solutions: $0<T<R=J$ *and* $0=T=R<J$

We obtain two possible corner solutions depending on the value of η: i) $\eta\geq\eta*$, and ii) $0<\eta\leq\eta_0$.

Case 1. $\eta\geq\eta*$. The following Proposition states the result.

Proposition 4. *Assume that* $\beta<0$, $\alpha>1$ *and* $\eta_0>0$: *if* $\eta\geq\eta^*$, *then* $T=T_{\max}(\eta^*)=\frac{J_{\eta_0}}{\eta^*}$ *and* $R=J$ *is the unique solution to Eqs. (30) and (31) such that* $0<T<J$.

Proof. See Appendix. □

Note, first, that if $\eta=\eta*$, interior and corner solutions coincide. And, second, that $T=R=J$ is also a solution. However, by using an indirect utility argument, this solution is dominated by $T=T_{\max}(\eta^*)=\frac{J_{\eta_0}}{\eta^*}$ and $R=J$. Leisure is zero in both cases ($R=J$) and so are pension benefits. But labor income is zero, and so is consumption, if $T=R$, while labor income and consumption are positive if $T=T_{\max}(\eta*)<R$.

Case 2. $0<\eta\leq\eta_0$. Assume $\eta=\eta_0$: from Eqs. (30) and (31) we obtain

$$R=\min\{T,J\},\ \text{and} \tag{40}$$

$$Tm(T)=\int_T^{\min\{T,J\}} m(\tau)\mathrm{d}\tau+\int_{\min\{T,J\}}^J \vartheta m(\tau)\mathrm{d}\tau. \tag{41}$$

Given that individuals never survive the age J, it must be the case that $T\leq J$ and, therefore, $R=T$ *always*. This implies that individuals never contribute to the social security and, therefore, ϑ *must be* 0 *if* the social security budget is balanced which must be the case along balanced growth paths. If this is the case, optimal T is given by (40), (41), $T=\min\{T, J\}$ and $\vartheta=0$ so that $Tm(T)=0$ which admits two solutions: $0<T=R=J$, where $T=T_{\max}(\eta_0)$, and $0=T=R<J$. The latter is preferred to the former. Consumption is zero in both cases: both labor income and pension benefits are zero because $T=R$ [recall Eqs. (14) and (15)]. But leisure is positive ($R<J$) in the latter case, and zero in the former ($R=J$).

Consider now that $0<\eta<\eta_0$. Given that $T\leq J$, from (30) one has that $R=T\eta/\eta_0<J$ which makes sense only if $R=T=0$. Otherwise, if $T>0$, one has that $R=T\eta/\eta_0<T$ and, therefore, $R<T$,

Table 1
Interior vs. corner solutions

$0<\eta\leq\eta_0$	$\eta_0<\eta<\eta^*$	$\eta=\eta^*$
$0=T=R<J$	$0<T<R<J$	$T=T_{\max}(\eta^*)=\frac{J\eta_0}{\eta^*}$ $R=J$

which makes no sense. Individuals would retire before completing education and entering the labor market. As before, ϑ must be 0 if the social security budget is balanced. Note that in *Case ii*) the economic meaning is completely absent: there would be no human capital, nor production, nor consumption.

The following Proposition states the result.

Proposition 5. *Assume that $\beta<0$, $\alpha>1$ and $\eta_0>0$. If $0<\eta\leq\eta_0$ and social security budget is balanced, then $\vartheta=0$ and $0=T=R<J$.*

Proof. See Appendix. □

Table 1 summarizes the results in Propositions 2–5. Note that for $0<\eta\leq\eta_0$ we have only considered the possibility of social security budget balance.

2.5. Aggregates

In this subsection we obtain the aggregates for consumption and human capital at time τ. To this end we weight individuals' decisions by the size of the surviving population in the living cohorts, and sum them up across birth dates.

- *Aggregate consumption*. From Eqs. (1) and (4) aggregate consumption can be expressed as, $C(\tau)=\int_{\tau-J}^{\tau}c(t,\tau)\zeta e^{nt}m(\tau-t)\mathrm{d}t$, where $c(t,\tau)$ represents consumption at τ of an individual born at t, and $\zeta e^{nt}m(\tau-t)$ denotes the measure of population of t-th generation still alive at time τ.
- *Aggregate human capital*. From Eqs. (1) and (4) we obtain aggregate human capital as

$$H(\tau)=\int_{\tau-R(\tau)}^{\tau-T(\tau)}h(t)\zeta e^{nt}m(\tau-t)dt, \tag{42}$$

where the last generation to enter the labor market was born at $\tau-T(\tau)$, and the last generation to retire from their jobs was born at $\tau-R(\tau)$. The cohort born at t and still in the labor force has a measure equal to $\zeta e^{nt}m(\tau-t)$, and their members have a stock of individual human capital $h(t)$, making this a *vintage* model.

Vintage models are often used both in economies with *physical* capital and in economies with *human* capital. In the first case the aggregate stock of capital installed in firms consists of capital goods of different ages, usually embedding different technologies (with more productive technologies in the more recent ones). In the second (*i.e.*, our) case, younger workers incorporate higher levels of human capital than their predecessors in a growing economy (although without their labor expertise which, for simplicity, we are not considering here).[10]

[10] Among the first type one could mention, *e.g.*, Jensen et al. (2001). And among the second type, *e.g.*, Boucekkine et al. (2002), Echevarría (2003) and Violante (2002).

From Eq. (42) we define *average human capital* at time τ, first introduced in the individual human capital production function Eq. (12), as

$$\bar{H}(\tau) = \frac{H(\tau)}{\kappa\zeta e^{nt}}, \tag{43}$$

where the denominator represents *total* population at time τ, defined in Eq. (5).

An indicator that can be used to explain growth in vintage models is the *quality* of human capital. This is given by the degree of obsolescence of the human capital which, in our case, is given by the *average tenure of active workers* L, which given the age distribution Eq. (7) is equal to

$$L(\tau) = \frac{\int_{T(\tau)}^{R(\tau)} [a - T(\tau)] e^{-na}(e^{-\beta a} - \alpha) \mathrm{d}a}{\int_{T(\tau)}^{R(\tau)} e^{-na}(e^{-\beta a} - \alpha) \mathrm{d}a}. \tag{44}$$

The numerical exercises in Section 3, however, will not show a monotonic relationship between L and γ.

2.6. Social security

Assuming that social security balances its budget on a period by period basis, the following equality must hold at each time τ

$$\int_{\tau-R(\tau)}^{\tau-T(\tau)} s\omega(\tau)h(t)\pi(t)m(\tau-t)\mathrm{d}t = \int_{\tau-J}^{\tau-R(\tau)} b(t)\pi(t)m(\tau-t)\mathrm{d}t. \tag{45}$$

The left-hand-side represents the social security tax revenue from active generations [*i.e.*, born after $\tau - R(\tau)$, but before $\tau - T(\tau)$]. The right-hand-side equals the pension benefits paid to retirees [*i.e.*, individuals born after $\tau - J$, but before $\tau - R(\tau)$].

2.7. Equilibrium

Definition 1. An equilibrium path for this economy is defined as a sequence of quantities $\{T(\tau), R(\tau), C(\tau), h(\tau), H(\tau), \bar{H}(\tau), Y(\tau)\}_{\tau}^{\infty}=0$ and prices $\{\omega(\tau), D(t,\tau)=m(\tau-t)\}_{\tau}^{\infty}=0$ such that i) consumers maximize utility taking the sequences of average human capital in the economy, wage per efficiency unit and discount factor, and the parameters representing the social security policy $\{s, \vartheta\}$ as given; ii) firms maximize profits taking the sequence of wage per efficiency unit as given; iii) the government chooses the replacement rate ϑ, for a given social security tax rate s such that social security budget is balanced at each instant; and iv) goods market clears.

In this article we only consider *balanced growth paths* characterized by the fact that *aggregate* variables $\{C(\tau), H(\tau), Y(\tau)\}$ grow at a constant rate or, equivalently, *per capita* variables $\{h(\tau), \bar{H}(\tau), w(\tau)\}$ grow at a constant rate γ. Moreover, variables indicating education time duration T and retirement age R are constant and, therefore, do not depend on the worker's birth date.

2.8. The balanced growth path

Definition 2. Balanced growth paths in this economy are defined as sequences of quantities $\{T, R, C(\tau), h(\tau), H(\tau), \bar{H}(\tau), Y(\tau)\}_{\tau}^{\infty}=0$ and prices $\{\omega, D(t,\tau)=m(\tau-t)\}_{\tau}^{\infty}=0$ such that i) conditions i)–iv) in Definition 1 are met, and ii) all aggregate variables in per capita terms grow at a constant rate γ.

In order to obtain the steady state growth in this economy, we first calculate the rate of growth of aggregate human capital: the sum of the population growth rate plus the growth rate of average human capital. To obtain the latter we substitute Eqs. (12) and (43) into Eq. (42),

$$\bar{H}(\tau)\kappa\zeta e^{n\tau} = \int_{\tau-R}^{\tau-T} \mu\bar{H}(t)T\zeta e^{nt}m(\tau-t)\mathrm{d}t.$$

If we take into account that along the balanced growth path $\bar{H}(t)=\bar{H}(\tau)e^{-\gamma(\tau-t)}$ must hold, and we make the following change of variable $z=\tau-t$, on recalling the survival probability Eq. (1) we can rewrite the above expression as,

$$\frac{\mu T}{\kappa}\int_{T}^{R} e^{-(\gamma+n)z}\left[\frac{\alpha-e^{-\beta z}}{\alpha-1}\right]\mathrm{d}z = 1, \tag{46}$$

which implicitly characterizes the per capita growth rate γ as a function of μ, T, κ, R, n, α and β. *Ceteris paribus*, from Eq. (46) one can see that for higher values of retirement age R or productivity in human capital production μ, the economy's long run per capita GDP growth γ will be higher. If the length of education T becomes longer, then an ambiguous result shows up: i) aggregate human capital is enlarged (individual human capital stocks are higher), but ii) the share of active population becomes smaller. Since the LHS of Eq. (46) is decreasing in γ, we have the following:

Proposition 6. *In the case with $0<T<R<J$, if there is a per capita growth rate γ that satisfies Eq. (46), it must be unique.*

Proof. Omitted. □

An open question is the convergence of this economy to the steady state. It can be shown that the dynamics of aggregate human capital is characterized by a second order delayed differential equation with delayed derivatives. Its resolution requires numerical methods, because no mathematical result that sets conditions on the parameters (T and R, in particular) for it to converge is known.[11]

Given that along balanced growth paths $R(\tau)=R$, $T(\tau)=T$, $\omega(\tau)=\omega$, and $\bar{H}(t)=\bar{H}(\tau)e^{-(\tau-t)}$, $\pi(t)=\pi(\tau)e^{-n(\tau-t)}$, taking into account Eqs. (12), (14) and (15), and after a change of variable $z=\tau-t$, the equation for social security budget balance Eq. (45) can be rewritten as

$$s\int_{T}^{R} e^{-(\gamma+n)z}m(z)\mathrm{d}z = \vartheta(1-s)\int_{R}^{J} e^{-(\gamma+n)z}m(z)\mathrm{d}z. \tag{47}$$

Finally, if the solution for T and R happens to be interior, the payroll tax rate s and the replacement rate ϑ which balance the social security budget are unique.

Proposition 7. *In the case with $0<T<R<J$, there is a unique pair of the payroll tax-replacement rates (s, ϑ) which satisfies the social security budget balance.*

Proof. Omitted. □

[11] Eq. (42) is exactly the same as the one obtained in Boucekkine et al.'s (2002), Eq. (23), p. 353.

Table 2
Benchmark case

Demographics	
Parameters	*Results*
η=0.010 (Id.),	$\hat{a}$=31.65 (35.9), $\bar{a}$=37.08 (36.2),
β=−0.017,	LE=77.93 (77), $1/\kappa$=0.019 (0.015),
α=7.5	J=118.5
Non-demographics	
Parameters with S. S.	*Results*
s=0.135 (Id.), μ=0.286,	T=34.5 (12.33), R=60.0 (61.6),
ϕ=0.485, ω=24.302	γ=1.97% (2.1%), ϑ=0.39
Parameters without S. S.	*Results*
s=0, μ=0.251,	T=28.6, R=60.0, γ=2.0%
ϕ=8.3075, ω=1.006	

Observed values are shown in parentheses.

To sum up, assuming interior solutions for T and R, along the balanced growth equilibrium path conditions Eqs. (33), (34), (46) and (47) must be met. This makes four non-linear equations in four unknowns: T, R, γ and ϑ (for a given s). Unlike Boucekkine et al. (2002), one cannot obtain analytically a relationship between T and in our model: γ influences T through social security budget balance which in turn affects ϑ; and T affects γ through the per capita GDP growth equation. Therefore, it is not possible to obtain a replica of their Proposition 3.4. [See Boucekkine et al., 2002, Proposition 3.4, p. 355.] Once the individual problem and the uniqueness of the steady state growth rate and the social security budget balance are solved analytically, all the results that follow in the next section are strictly numerical.

3. A numerical example

3.1. Calibration

In this section we give values to the basic parameters to calculate numerically the steady state equilibrium for a benchmark case. In order to illustrate the role played by social security in our model, we run a first experiment in which we compare the responses of this economy to exogenous changes in life expectancy under two alternative scenarios: with and without social security. To this end, and taking some features of the U.S. economy as an example, we use two sets of parameters which give us, with some approximation, the same benchmark steady state.[12] The summary of the two sets of parameters, the steady state theoretical values for the benchmark model and the empirically observed values are shown in Table 2. In a second experiment, we analyze the effects of changes in social security policy.

Concerning our first experiment, we choose one set of demographic parameters α, β and n so that we are able to approximate the observed mean and median ages and life expectancy at birth, $\bar{a}$, $\hat{a}$ and LE, respectively. The crude birth rate $1/\kappa$ turns out to take a

[12] At any rate, the parameter sets must be such that $\eta_0 < \eta < \eta^{**}$. If $\eta \leq \eta_0$, condition in Eq. (46) is not met; and if $\eta \geq \eta^*$, from Eq. (47) one would have that ϑ would be infinite. Therefore, we focus on interior solutions for T and R.

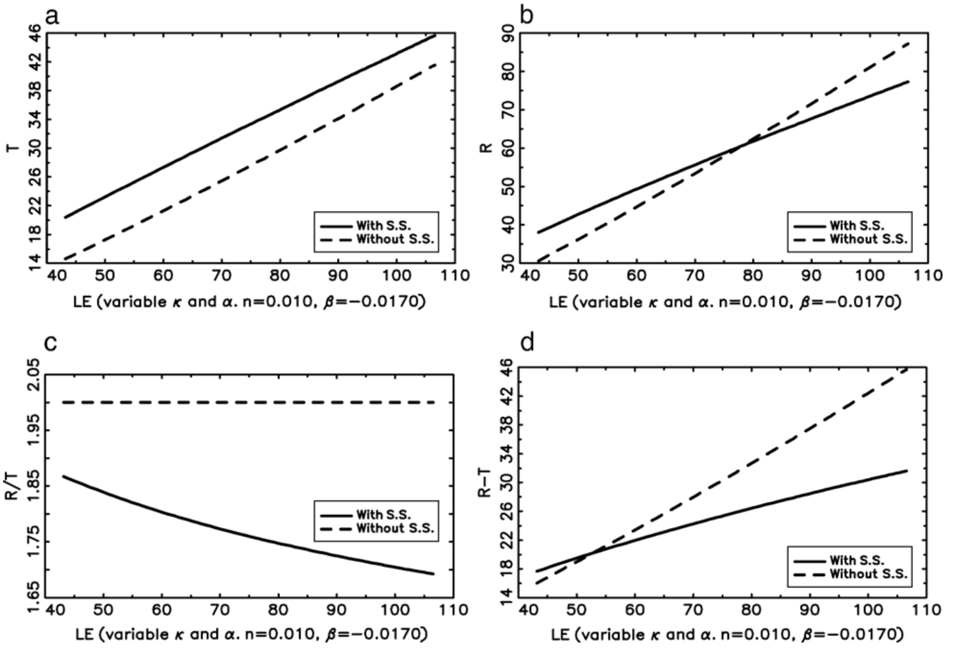

Fig. 1. a: Education vs. Life Expectancy with and without social security. b: Retirement vs. Life Expectancy with and without social security. c: Retirement/Schooling vs. Life Expectancy with and without social security. d: Active Life vs. Life Expectancy with and without social security.

reasonable value too.[13] As for the non-demographic parameters, we choose two values for each of μ, ϑ and ω in such a way that the resulting T, R and γ obtained with ($s>0$) and without ($s=0$) social security satisfy two conditions: reasonably close between them and also to observed values, in particular in R and γ.[14]

Once we have set up the benchmark case, the experiment consists in generating a range of values for life expectancy (between 40 and 100). There are 4 ways of modelling increases in life expectancy depending on, first, whether they are due to a lower mortality rate for young or for

[13] Data on n have been obtained from *U.S. Census Bureau, National and Population Estimates*, NST-EST2003-pop-chg, Annual Estimates of the Population Change for the United States, and Puerto Rico and State Rankings: July 1, 2002, to July 1, 2003, p.1, available at http://www.census. gov/popest/states/NST-EST2003-pop-chg.html. For LE see *2004 World Population Data Sheet of the Population Reference Bureau*, Demographic Data and Estimates for the Countries and Regions of the World, p. 7, available at http://www.prb.org. For $\bar{a}$ and $\hat{a}$ see *U.S. Census Bureau, National Population Estimates*, Characteristics, Annual Estimates of the Population by Sex and Five Year Age Groups for the United States: April 1, 2000 to July 1, 2003 (NC-EST2003-01) p.1, available at http://www.census.gov/popest/national/asrh/NC-EST2003/ Finally, for data on $1/\kappa$ see *Vital Statistics*, in *U.S. Census Bureau*, *Statistical Abstract of the United States: 2004–2005*, Table 72, p. 61.

[14] The value of s has been obtained from Coronado et al. (2000), p. 10. For data on T see Butcher and Case (1994). Bassanini and Scarpetta (2001), p. 28, show the increasing time trend of average years of education among working population for 21 OECD countries between 1971 and 1978. As for retirement age R see, *e.g.*, Gendell (1998). For γ see *Income, Expenditures and Wealth* in *U.S. Census Bureau, Statistical Abstract of the United States: 2004–2005*, Table 648, p. 430. Finally, as for the replacement rate, observed values vary substantially depending on the worker's individual characteristics, See, *e.g.*, Diamond and Gruber (1997) or Kotlikoff et al. (1999).

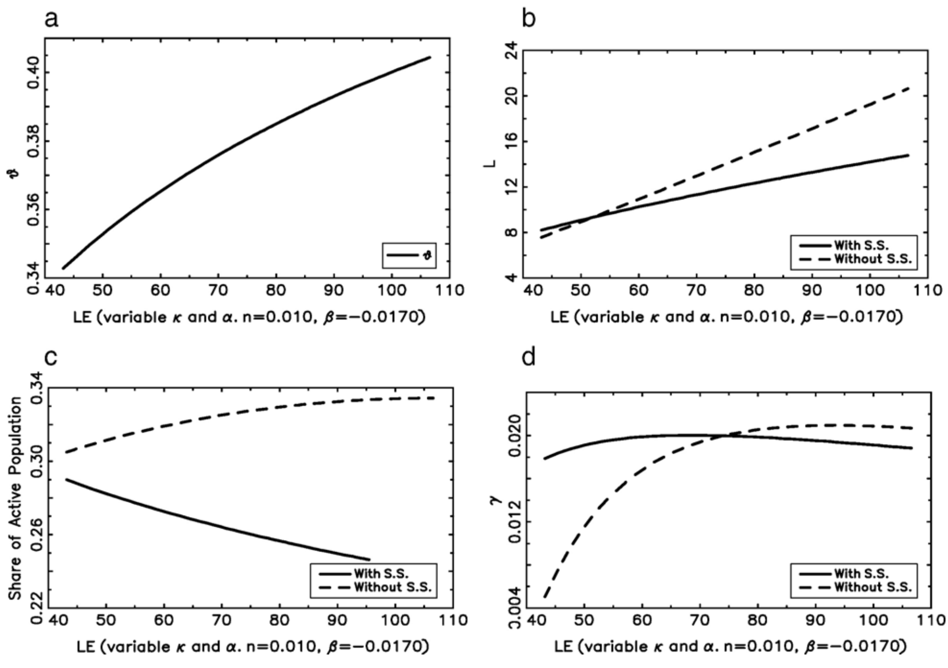

Fig. 2. a: Replacement Rate vs. Life Expectancy. b: Average Tenure vs. Life Expectancy with and without social security. c: Active Population vs. Life Expectancy with and without social security. d: Per Capita Growth vs. Life Expectancy with social security.

old agents (higher α or higher β, respectively); and, second, whether the population growth rate or the crude birth rate adjusts to mortality changes (constant n and lower $1/\kappa$ or lower n and constant $1/\kappa$, respectively). Here we focus on increases in life expectancy caused by lower mortality rates for young agents and lower crude birth rates. Next we discuss how our results would differ if changes in β or n had been considered. The results are illustrated in Figs. 1 and 2. We run an additional experiment: keeping the rest of the parameters constant, we analyze the response of our theoretical economy to changes in the social security tax rate s. The results are illustrated in Fig. 3.

3.2. Findings

In our model the engine of growth is given by the change in the average human capital of the economy. Average human capital, in turn, depends on individual decisions such as the optimal length of education and the optimal age of retirement which (in the case of T) affects not only their own productivity, but also the share of active workers in the population as T and R represent entry to and exit from the labor market, respectively. In addition, demographic characteristics, such as the survival rate distribution also impact average human capital. Thus, a decline in the mortality rate (*i.e.*, increases in life expectancy) implies both a *behavior effect* (through changes in T and R) and a *composition effect* (through changes not only in the age distribution of workers, but also on the range of active ages) which in turn imply changes in the growth rate. We argue below that both effects differ depending on whether there is unfunded social security in the economy or not.

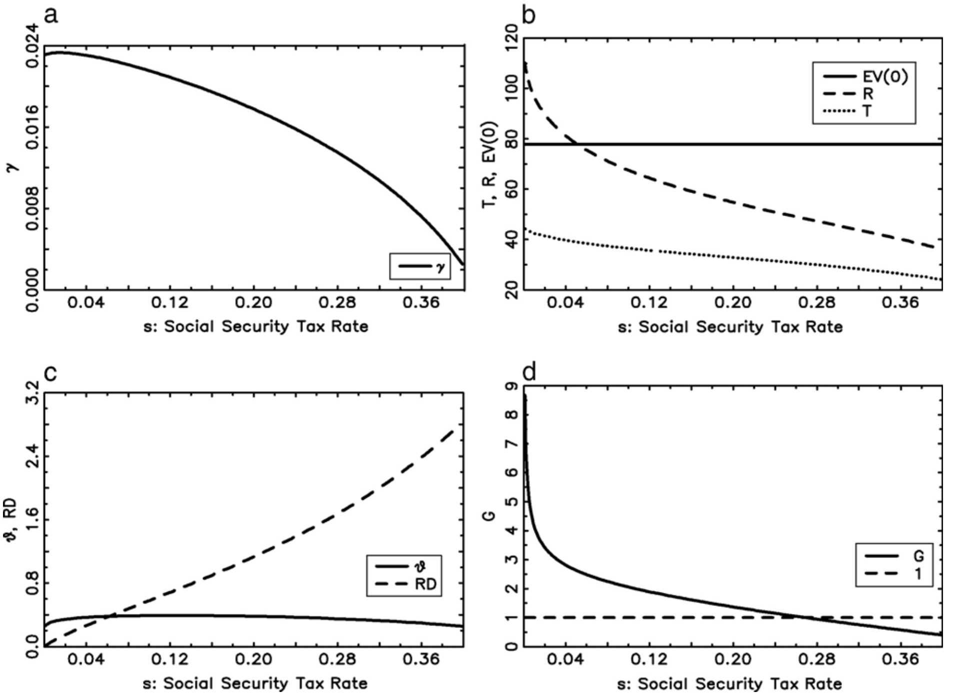

Fig. 3. a. Per capita growth rate. b. Life expectancy, education and retirement. c. Replacement rate and dependency ratio. d. Generosity of pensions.

We graph the response of individual and aggregate variables (in Figs. 1 and 2, respectively) to increases in life expectancy under both scenarios, *i.e.* with and without social security.[15]

As life expectancy goes up, the length of education time increases in both scenarios. Higher survival probabilities increase the expected flow of future wages, thereby giving incentives to increase the length of education time. In Eq. (31) the left-hand side represents the marginal cost of increasing T, and the two terms on the right-hand side represent the marginal benefit, *i.e.* the expected flow of future wages (pension benefits included). Without social security only the first term is present. With social security (*if* pensions depend on past contributions), however, an additional term shows up as the incentives to a higher T increase [see Fig. 1a].

Concerning the retirement age decision, we can see in Eq. (33) that increases in life expectancy affect R through *both* its effect on T (as in an economy without social security) and the social security policy represented by the replacement and the contribution rates ϑ and s, respectively (which enter the definition of η_0). Therefore, with social security the relationship between T and R is not proportional. In an economy with no social security, however, increments in life expectancy give rise to increases in R proportional to those in T [see Fig. 1c]. Our numerical experiment shows that increases in life expectancy allow greater generosity of social security because ϑ becomes higher. Why? Workers are more productive (so that they contribute more) and their survival probabilities are higher (particularly among young agents) [see Fig. 2a]. Therefore,

[15] The range of values for α goes from 3.4 to 13.5; n and β take on the same values as in the benchmark case, 0.010 and -0.0170, respectively. As a result, LE ranges between 43.2 and 106.5, and κ^{-1} between 0.028 and 0.015.

(unfunded) social security gives incentives to early retirement, something that is well known in the literature. This explains why as life expectancy increases, the retirement age increases at a lower rate than in an economy with no social security. In fact, for high enough levels of life expectancy, R is lower with than without social security [see Fig. 1b]. Along these lines, Fig. 1c shows the ratio of R to T under both regimes: without social security it is just a constant; with social security, however, it is decreasing in T.

This lower increment in R implies that the active life length $R-T$ increases at a lower rate with than without social security. So, even though the increment in T is expected to be higher in the presence of social security (as discussed above), it is lower than what it would have been had retirement age been exogenous [see Fig. 1d].

We next graph the response of aggregate variables to increases in the life expectancy with and without social security in Fig. 2. As advanced above, increases in life expectancy bring about higher generosity of social security represented by a higher ϑ [see Fig. 2a].

Along a balanced-growth path economy, changes in active life parallel changes in average tenure of active workers, L [see Fig. 2b]. Therefore, it is not surprising that L increases along with life expectancy, but at a lower rate in the presence of social security than in its absence, thus displaying a similar pattern to the one of $R-T$. This has a composition and a behavior (productivity) effect on the per capita growth rate of the economy. *First*, the share of active workers in the population falls. Why? Even though the span of active life $R-T$ becomes larger, the whole population age distribution changes: it can be shown that the weight of young individuals falls relatively, while the weight of older individuals becomes relatively higher. However, in an economy without social security the proportion of active workers *increases* with life expectancy. This composition effect turns out to be crucial for understanding the response of per capita growth as we see next [see Fig. 2c].[16] *Second*, given the lower increment in average tenure, the obsolescence of active workers' human capital will increase at a lower rate too.

We finally arrive at the relationship between life expectancy and per capita GDP growth, and find that it exhibits an inverted U pattern under both scenarios. In the social security regime, however, the negatively sloped part shows up at much lower levels of life expectancy. The explanation must be sought in the incentives to early retirement that social security introduces and that, as we have just seen, make the share of working population fall substantially [see Fig. 2d].

More precisely, without social security the vintage characteristic appears to play an important role in explaining the decreasing part of the inverted U: if the mortality rates fall, the proportion of workers whose schooling took place a long time ago (and who have become obsolete) is higher. With social security, however, the vintage characteristic does not play such an important role, and the explanation must be sought in the decrease in the share of workers: if the rate of mortality falls, the per capita GDP growth may end up decreasing simply because workers retire earlier.

How would our results have changed if we had allowed reductions in the population growth rate n (instead of the crude birth rate $1/\kappa$)? Or in the mortality for old agents through increments in β (instead of the mortality for young agents through increments in α)? A declining crude birth rate along with an increasing life expectancy and a constant population growth rate are closer to present-day observed demographic characteristics than constant crude birth rates and increasing population growth rates. In terms of the model, the results (not shown for reasons of space)

[16] It can be shown that in the social security case, increases in life expectancy are accompanied by increments in the dependency ratio (retirees to workers ratio), a well documented fact in real economies. [See, *e.g.*, Diamond and Gruber (1997).] In the no social security case, however, increments in life expectancy accompany falls in the dependency ratio.

change only quantitatively. Regarding whether the reduction in the mortality rates affects mainly young or old individuals, Kalemli-Ozcan (2002b) claims that during the last two centuries life expectancy has doubled in most parts of the world mostly due to larger declines in child rather than adult mortality.[17] It can be shown that the main difference shows up in the share of workers. If increments in β rather than in α are assumed, the share of workers in an economy without social security *decreases*: the increase in the population size mainly takes place at older ages.

How do these results match observed facts? Or, does the existence of unfunded social security affect the response of the economy to changes in life expectancy? In the working paper version of this article we continue with the U.S. case and split the data into two periods, 1870–1940 and 1950–2000, which can represent our two regimes: without and with social security, respectively.[18] Regarding the first period, we observe what our model predicts that increased life expectancy is accompanied by three facts: i) increased average years of education, ii) an inverted *U* pattern for the average annual growth rate of per capita GDP, and iii) a fairly stable labor force participation. For the second period, the predictions of our model partly resemble observed facts: as before, augmented life expectancies go along with facts i) and ii). As for the observed *aggregate* (*i.e.*, for both sexes) labor force participation, it exhibits an *increase* as opposed to the decline that our *one-sex* model predicts. Why? A major distinction between *male* and *female* labor force participation rates must be made and in particular among younger and older workers. Whereas participation rates among women have increased substantially, men's have declined: for those aged 16 to 24, the decline likely reflects increases in *T*; and for the elderly, the reason may be sought in a higher availability of pensions and disability awards, *i.e.*, social security. Our model does not predict a decline in *R* in the social security regime, but it does predict a *lower* increment in the retirement age. Perhaps an extended number of observations would allow us to observe a deeper reduction in the per capita GDP growth rate and a fall in aggregate participation rates as observed among male workers.

3.2.1. Social security

Fig. 3 shows the effects of changes in the social security contribution rate *s* from 0.1% to 40%. The first result is a net discouraging effect upon human capital accumulation and retirement age.

A higher social security tax rate *s* reduces the net wage $w(t)=(1-s)\omega\mu\bar{H}(t)T$ [for a given $\bar{H}(t)$], so that incentives to devote a fraction of one's lifetime to education *T* are reduced.[19] As noted in subsection 2.4.1, $R=\eta(1-\vartheta)(1-s)\omega T$ so an increase in *s* results in a drop in *R*. Given that the replacement rate ϑ remains relatively unchanged, pension benefits fall as well, so that incentives to lengthen the education phase are reduced even more, which generates an even larger reduction in *R*.

Given the reaction of *R*, it is easy to understand the response of the dependency ratio, RD: the higher the social security contribution rate, the higher the dependency ratio, and the lower the share of active workers. This result is in line with Coile and Gruber (2000) and Kalemli-Ozcan (2002a), among others.

[17] Zhang et al. (2001) claim that during the early stages of mortality falls these concentrate on the younger population; but as mortality keeps going down from low enough levels, most additional years are gained after retirement age.

[18] The detail of the results is not shown here for space saving. See Echevarria *et al.* (2005).

[19] In Zhang and Zhang (2004) the effect is just the opposite, but the mechanism is different: parents pay for their children's education. Higher pay-roll taxes reduce the net wages that children will obtain and increase parents' incentives to spend more on their children's education (and have less children).

Once the behavior of T, R and RD has been analyzed, the performance of γ is as expected: higher social security contribution rates accompany lower rates of per capita GDP growth.[20]

Finally, focusing on the replacement rate, ϑ remains hardly changed. How is it possible that (upon increasing s) RD goes up and ϑ stays almost constant? Remember, first, that ϑ represents the replacement rate defined on *net* wages $(1-s)w$; and, second, as noted in the beginning of this subsection, the social security budget balance condition depends on the dependency ratio, the contribution and the replacement rates *and also* the per capita growth rate of the economy γ.

As a result, one would expect the *generosity* of the pension scheme to fall. One way to measure this consists in calculating the ratio of the sum of present values of expected pension benefits to the sum of present values of expected social security contributions. Thus, from Eqs. (1), (25), (12), (13), (14) and (15), and recalling that $\omega(\tau)=\omega$, one has

$$G=\frac{\int_{t+R}^{t+D} D(t,\tau)b(t)\mathrm{d}\tau}{\int_{t+T}^{t+R} D(t,\tau)s\omega h(t)\mathrm{d}\tau}=\frac{\vartheta(1-s)\left[\alpha(J-R)+e^{-\beta J}-e^{-\beta R}\right]}{s\left[\alpha(R-T)+e^{-\beta R}-e^{-\beta T}\right]}. \tag{48}$$

The relationship between G and s that we obtain is strictly decreasing. In fact, for low enough s (in our numerical example $s<0.27$), the G that we obtain is higher than 1 (*actuarially more than fair* pension benefits).

4. Conclusions

In this article we have studied the relationship between life expectancy and per capita GDP growth rate. We have used a vintage growth model with a pay-as-you-go social security system where individuals choose education time and retirement age and where pensions depend on the contributions made by workers during their active lives. This way the flow of income during the retirement period also depends on the education time investment.

The results obtained in the first part of the article are analytical. We characterized the individual's parameter space which establishes the type of solution for length of education and retirement age. We also proved the existence of, at most, one steady state per capita growth rate and of one unique steady state budget balance for social security. In the second part we compared the responses of the economy to exogenous changes in life expectancy under two regimes (with and without social security), obtaining numerical results.

In our model the engine of growth is given by the change in the average human capital of the economy. Average human capital depends on, first, individual decisions such as optimal schooling and optimal age of retirement which affect their own productivity and the share of active workers in the population; and, second, demographic characteristics, namely the survival rate distribution. We saw that increases in life expectancy imply both a *behavior effect* (through changes in schooling and retirement age) and a *composition effect* (through changes in the age distribution of workers and in the range of active workers) which in turn imply changes in the growth rate. We found that in an economy with no social security the vintage characteristic seems to play a relevant role as the proportion of agents whose schooling took place a long time ago becomes higher with higher levels of life expectancy. However, in an economy with social

[20] Nevertheless, the values of s for which γ attains the maximum is not zero, but slightly positive: in our numerical example, we obtain that $\gamma=2.3\%$ for $s=1.4\%$. The existence of externalities in the human capital accumulation explains why positive social security may promote growth.

security, the vintage description of the economy does not play such an important role in explaining the decreasing part of the life expectancy-per capita growth locus. In this case, the decrease in the share of workers as life expectancy goes up is the main factor.

Finally, we studied the relationship between the size of social security and the per capita GDP growth rate. We found that such a relationship is mostly negative, except for very low values for the social security contribution rate. The explanation lies in the discouraging effect that social security imposes on education and, in particular, retirement age, which causes a fall in the share of the working population in the economy.

We believe that this line of research requires further empirical work, especially in western economies in which life expectancy has reached significantly high levels and where there is strong debate about the sustainability of current unfunded social security systems and the convenience of postponing retirement age.

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Appendix A.

Proof of Lemma 1. Substituting $x=0$ into Eq. (38) one has that $M(0,\eta)=\frac{\vartheta}{\beta}[1-\alpha(1-\ln\alpha)]$, where $1-\alpha(1-\ln\alpha)>0$ if $\alpha>1$. □

Proof of Lemma 2. This follows 8 steps (S1–8) whose respective proofs are sketched. Assume $\beta<0$, $\alpha>1$ and $\eta_0>0$.

S1. *If $\eta<\eta_0$, then $K(\eta)$ is strictly increasing*. From Eq. (39) one obtains that $K'(\eta)=\frac{\eta_0\ln\alpha}{\eta^2\beta}[2\alpha-\alpha^{\frac{\eta_0}{\eta}}(2+\frac{\eta_0}{\eta}\ln\alpha)]=\frac{\eta_0\ln\alpha}{\eta^2\beta}g(\alpha,\eta)$, where $g(\alpha,\eta)\equiv2\alpha-\alpha^{\frac{\eta_0}{\eta}}\left(2+\frac{\eta_0}{\eta}\ln\alpha\right)$. Given the assumptions on α, β and η_0, $K'(\eta)>0\Leftrightarrow g(\alpha,\eta)<0$, which can be proven with some algebra.

S2. *The function $K(\eta)$ satisfies that i) $K(\eta_0)=0$, and ii) $K'(\eta_0)>0$.* i) Trivially, from η_0 and Eq. (39). ii) From S1 one already has $K'(\eta)$. Some algebra gives the result.

S3. *The function $K(\eta)$ is negative for all $\eta<\eta_0$.* From S1 and part i) in S2, it is trivial.

S4. *The function $K(\eta)$ satisfies that i) $K(2\eta_0)>0$, and ii) $K'(2\eta_0)<0$. i)* It can be shown from Eq. (39) that $K(2\eta_0)=\frac{\alpha^{1/2}}{\beta}\left(\frac{\ln\alpha}{2}+1-\alpha^{1/2}\right)=\frac{\alpha^{1/2}}{\beta}i(\alpha)>0\Leftrightarrow i(\alpha)<0$, which is the case if $\alpha>1$. ii) Evaluating $K'(\eta)$ from S1 at $\eta=2\eta_0$, one has $K'(2\eta_0)=\frac{\ln\alpha}{4\beta\eta_0}j(\alpha)$, where $j(\alpha)\equiv2\alpha-\alpha^{\frac{1}{2}}\left(2+\frac{\ln\alpha}{2}\right)$. Given the restrictions upon β and η_0, then $K'(2\eta_0)<0\Leftrightarrow j(\alpha)>0$, which is the case for $\alpha>1$.

S5. *One unique $\hat{\eta}\in(\eta_0,2\eta_0)$ exists such that $K'(\hat{\eta})=0$.* From S1 one obtains that $K'(\eta)=\frac{\eta_0\ln\alpha}{\beta\eta^2}g(\alpha,\eta)$. Thus, $K'(\eta)=0\Leftrightarrow g(\alpha,\eta)=0$. Moreover, $g(\alpha,\eta)=0\Leftrightarrow l(\alpha,\eta)=m(\alpha,\eta)$, where $l(\alpha,\eta)\equiv2\alpha^{\frac{\eta-\eta_0}{\eta}}$ and $m(\alpha,\eta)\equiv2+\frac{\eta_0}{\eta}\ln\alpha$. It is shown that there is one unique $\hat{\eta}\in(\eta_0,2\eta_0)$ such that $l(\alpha,\hat{\eta})=m(\alpha,\hat{\eta})$, *i.e.*, $g(\alpha,\hat{\eta})=0\Leftrightarrow K'(\hat{\eta})=0$.

S6. *If $\eta>2\eta_0$, then $K(\eta)$ is strictly decreasing*. From S1 one has that $K'(\eta)=\frac{\eta_0\ln\alpha}{\beta\eta^2}g(\alpha,\eta)$. We have proved in S4 that $K'(2\eta_0)<0$, *i.e.*, $g(\alpha,2\eta_0)>0$. Given $\partial g(\alpha,\eta)/\partial\eta>0$, if $\eta>2\eta_0$, then $g(\alpha,\eta)>0$ and $K'(\eta)<0$.

S7. *The function $K(\eta)$ satisfies that* $\lim_{\eta\to\infty}K(\eta)<0$. From Eq. (39) one has that $\lim_{\eta\to\infty}K(\eta)=\frac{\alpha}{\beta}\left(\ln\alpha+\frac{1}{\alpha}-1\right)\equiv\frac{\alpha}{\beta}o(\alpha)<0$ because $o(\alpha)>0$ for $\alpha>1$.

S8. *The function* $K(\eta)$ *satisfies that i) there is a unique* $\eta^* > 2\eta_0$ *such that* $K(\eta^*)=0$, *and ii)* $K(\eta)<0$ *for all* $\eta>\eta^*$. From S4, S6 and S7 the proof is trivial. From S3–5 and S8 the proof of Lemma 2 is trivial. □

Proof of Proposition 1. From Lemmas 1 and 2 the result is trivial. □

Proof of Proposition 2.[21] (Sketch) Assume that $\beta<0$, $\alpha>1$ and $\eta_0>0$. From Proposition 1 one has that there is at least one $x\in(0, T_{\max}(\eta))$ such that $M(x,\eta)=0$. To prove that it is unique, we split $M(x,\eta)$ as the difference of two functions $p(x,\eta)\equiv 2\alpha x - \frac{1-\vartheta}{\beta}e^{-\beta\frac{\eta}{\eta_0}x} - \frac{\vartheta}{\beta}e^{-\beta J}$ and $q(x,\eta)\equiv xe^{-\beta x} + \vartheta\alpha J + (1-\vartheta)\alpha x\frac{\eta}{\eta_0} - \frac{1}{\beta}e^{-\beta x}$, and prove that one unique intersection point exists between them for some $x\in(0, T_{\max}(\eta))$ such that $p(x,\eta)=q(x,\eta)\Leftrightarrow M(x,\eta)=0$. □

Proof of Proposition 3.[22] (Sketch) Given Eq. (1), $x=T$ is a solution to Eq. (31) if and only if $\hat{M}(x,\eta)=0$, where

$$\hat{M}(x,\eta)\equiv x(\alpha - e^{-\beta x}) + \alpha\left[x - \min\left\{\frac{\eta x}{\eta_0}, J\right\}\right] + \frac{e^{-\beta x} - e^{-\beta\min\left\{\frac{\eta x}{\eta_0}, J\right\}}}{\beta} + \vartheta\alpha\left[\min\left\{\frac{\eta x}{\eta_0}, J\right\} - J\right] + \frac{\vartheta\left[e^{-\beta\min\left\{\frac{\eta x}{\eta_0}, J\right\}} - e^{-\beta J}\right]}{\beta}. \tag{49}$$

The strategy of the proof follows these steps. First, we prove that, given the necessity for the existence of interior solutions that $\eta>\eta_0$, and $\hat{M}(J,\eta)=0$. Second, we prove that for values of x less than and close enough to J we will have that and $\hat{M}(x,\eta)>0$. Third, we prove that $\hat{M}(0,\eta)<0$. Fourth, given that $\hat{M}(x,\eta)$ is continuous, at least one $x\in(0,J)$ exists such that $\hat{M}(x,\eta)=0$. And fifth: given Eq. (36), from Eqs. (38), (39) and (49) it can be obtained

$$\hat{M}[T_{\max}(\eta),\eta] = M[T_{\max}(\eta),\eta]\equiv K(\eta).$$

If such an x in the forth step is unique and interior, $0<x<T_{\max}(\eta)<J$, then $\hat{M}[T_{\max}(\eta),\eta]=K(\eta)>0$, which implies that $\eta\in(\eta_0,\eta^*)$ as we have proven in Lemma 2. □

Proof of Proposition 4. (Sketch) The strategy of the proof is as follows. Consider $\beta<0$ and $\alpha>1$. Assume first that, $\eta=\eta_*>\eta_0>0$. First, from Lemma 2 one has that $M[T_{\max}(\eta_*), \eta_*]=0$; equivalently, $T_{\max}(\eta_*)\equiv\frac{J\eta_0}{\eta^*}<J$ is a solution to Eq. (31). And from Eq. (30) one has that $R=J$. In sum, $T=T_{\max}(\eta_*)\equiv\frac{J\eta_0}{\eta^*}$ and $R=J$ satisfy Eqs. (30) and (31), so that corner and interior solutions coincide. Second, we prove that it is the unique one which meets the condition $0<T<J$. To this end prove that $Tm(T)=\int_T^J m(\tau)d\tau\Leftrightarrow r(T,\alpha,\beta)\equiv\left(T-\frac{1}{\beta}\right)(\alpha-e^{-\beta T})-\alpha(J-T)=0$ has one unique solution $T<J$ for what the concavity of $r(T,\alpha,\beta)$ with respect to T is needed.

Assume now that $\eta>\eta_*>\eta_0>0$. In this case, this is the strategy of the proof. First, we show that there is no $x\in[0,T_{\max}(\eta)]$ such that Eq. (31) has a solution or that, equivalently, $\hat{M}(x,\eta)$ defined in Eq. (49) takes the value zero. According to Step 8, if $\eta>\eta_*$, then $K(\eta)<0$. Therefore, $\hat{M}[T_{\max}(\eta),\eta]<0$. $\hat{M}(x,\eta)$ can be rewritten as $\hat{M}(x,\eta)=\hat{p}(x,\eta)-\hat{q}(x,\eta)$, where $\hat{p}(x,\eta)\equiv 2\alpha x-(1-\vartheta)\beta^{-1}\exp(-\beta\min\{\frac{\eta x}{\eta_0},J\})-\vartheta\beta^{-1}e^{-\beta J}$ and $\hat{q}(x,\eta)\equiv xe^{-\beta x}+\alpha\vartheta J+\alpha(1-\vartheta)\min$

[21] We owe the last part of the proof to Águeda Madoz, our research assistant.

[22] We believe that there is a mistake in the proofs of Lemma 2.2 and Lemma 2.3 in Boucekkine et al. (2002). The authors claim that "Trivially, $\lim_{x\to+\infty} M(x)=+\infty$ …", but this is not true. [See proof of Lemma 2.2 on page 367, and proof of Lemma 2.3 on page 368.] Notice that $m(x)$ as defined in Eq. (1) is identically equal to 0 for $x\geq J$. Nevertheless, this does not affect the validity of their results.

$\{\frac{\eta x}{\eta_0}, J\} - \frac{e^{-\beta x}}{\beta}$. One just need to prove that $\hat{p}(x,\eta)$ and $\hat{q}(x,\eta)$ do not cross each other at any $x \in [0, T_{\max}(\eta)]$; equivalently, no $x \in [0, T_{\max}(\eta)]$ exists such that $\hat{M}(x,\eta)=0$. And if there is some x for which $\hat{M}(x,\eta)=0$, then $x \in (T_{\max}(\eta), J]$. Second, we show that $\hat{M}(J,\eta)=0$, therefore J is a solution. Third, we prove that one unique $x \in (T_{\max}(\eta), J)$ exists such that $\hat{M}(x,\eta)=0$, where $x = T_{\max(\eta_*)} \equiv \frac{J\eta_0}{\eta_*}$, so that $T = T_{\max(\eta_*)} \equiv \frac{J\eta_0}{\eta_*}$, and $R=J$. From $M(x,\eta)$ defined in Eq. (49), for $\eta > \eta_* > \eta_0$ and for x close enough to J, $\min\left\{\frac{\eta x}{\eta_0}, J\right\} = J$. Therefore, for x close enough to J, $\hat{M}(x,\eta)>0$. Thus, continuity of $\hat{M}(x,\eta)$ assures us that there exists *at least one* $T \in (T_{\max}(\eta), J)$ such that $\hat{M}(T,\eta)=0$. If $T > T_{\max}(\eta)$, then $R = \min\left\{\frac{T\eta}{\eta_0}, J\right\} = J$. But if so, from Eqs. (30) and (31) one has that T must satisfy $Tm(T) = \int_T^J m(\tau)\mathrm{d}\tau$. And this is, precisely, the case of $\eta = \eta^*$ studied above. □

Proof of Proposition 5. Assume that $\eta = \eta_0$: from Eq. (30) one has that $R = T \leq J$, so that social security tax revenues are zero. Social security budget balance is required so that $\vartheta = 0$. From Eqs. (40) and (41) we obtain that $Tm(T)=0$. The strategy of the proof consists of two steps. First, we prove that $Tm(T)=0$, which can be rewritten as $u(T) \equiv T(\alpha - e^{-\beta T})=0$ admits only two solutions, $T=0$ (so that $0=T=R<J$) and $T=J$ (so that $0<T=R=J$). And, second, we obtain the indirect utility function $V(T,\eta) = -\bar{H}\phi^{-1}\int_0^T \tau(e^{-\beta\tau}-\alpha)(1-\alpha)^{-1}\mathrm{d}\tau$, decreasing in T. Therefore, solution $0=T=R<J$ is preferred to solution $0<T=R=J$. □

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